SPECULARITY IN ALGEBRA

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The purpose of this article is to propose a partial answer to two questions raised by Mason and Pimm (1984) about a quarter of a century ago. The questions are:

> How can you expose the genericity of an example to someone who sees only its specificity? Apart from stressing and ignoring, and repeating the general statement over and over, how can the necessary act of perception, of seeing the general in the particular, be fostered?
>
> How can you discern the extent of the generality perceived by someone else when looking at a particular example together? (p. 287)

These two questions are pertinent to all areas of mathematics. For example, all school teaching of arithmetic processes (*e.g.*, how to add two digit numbers) is done through particular examples intended to be general. In this article, however, I will confine my discussion to the realm of the learning and teaching of algebra, where the act of "seeing the general in the particular" is practiced extensively and explicitly. Indeed, this often quoted phrase underlies the proposition that "much of algebra builds on students' extensive experiences with number" (NCTM, 2000). However, the route to algebra is not easy. Downgrading students' propensity to calculate (Küchemann & Hoyles, 2009), overcoming the reliance on empirical arguments (Healy & Hoyles, 2000), getting accustomed to seeing the structural features (Radford, 2010a) and perceiving algebraically useful patterns (Lee, 1996) are just a few challenges in students' passage from number to algebra. And it is a more demanding task for teachers, researchers or curriculum developers to foster it. Lee (1996, p. 95) points to this challenge when she shows how, in work conducted with David Wheeler, the interviewer's perception of pattern can interfere with the students' progress in pattern generalizing activities. This observation also shows how difficult it is, even in a supportive experimental setting, to find a general answer to Mason and Pimm's second question. However, as philosophers say, one way to answer a question is to replace it with a new (and hopefully better) question. This is what I attempt to do in this article.

First, I introduce two kinds of generalization, as distinguished by Polya (1945). Second, I exemplify a well-known algebraic generalizing task in the light of the two types of generalizations. Third, I explain a number of possible actions in response to students' difficulties with the task. The next step is "to withdraw from the immediate action" and discern what is worth noticing, "labelling it in some way so that it has a chance of coming-to-mind in the future" (Mason, 2010, p. 3). The approach used is the most mundane one: stressing and ignoring (Mason & Pimm, 1984) while exemplifying certain algebraic situations in which one of Polya's kinds of generalization is involved. Finally, the original questions are replaced by more subtle (and in a sense, more practical) ones, the aim of which is to direct attention to "noticing more sensitively and/or having alternative actions come-to-mind in the midst of preparing or conducting lessons or some other relevant setting" (Mason, 2010, p. 9).

Two kinds of generalization

Generally speaking, generalization occurs in two characteristically different ways:

“generalization is passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one” (Polya, 1945, p. 108). In the first type of generalization we generalize when we move from “one” to “any”; in the second type of generalization we generalize when we remove a *restriction*. The tension between the two types of generalizations is present in most algebraic tasks. For example, consider the following number trick expressed in the fashion of Sawyer (1943):

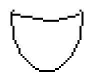
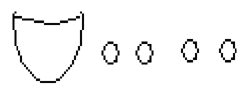
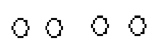
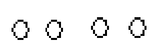
	Words	Picture	Algebra
I.	Think of a number.		n
II.	Add 4 to it.		$n+4$
III.	Take away the number you first thought of.		$(n+4)-n$ or 4
IV.	The answer is 4.		$(n+4)-n=4$

Table 1. Each bag is supposed to contain as many marbles as the number you thought of, whatever that was.

Passing from the number of marbles in my bag to “as many marbles as the number you thought of, whatever that was” (Sawyer, 1943, p. 81), I have successfully done the first type of generalization. Accordingly, *n* in the identity $(n + 4) - n = 4$ is *naturally* restricted to natural numbers. However, with the introduction of new kinds of numbers we may remove that restriction and see that the identity also holds for these new numbers; this is the second type of generalization. Obviously, the bag of marbles is not a useful tool for the second type of generalization. In the next section, I argue that the use of such "transitional" language for the first type is not as straightforward as it seems.

Transitional language

The use of "a kind of 'transitional' language prior to the standard alphanumeric-based algebraic language" (Radford, 2000, p. 239) is an old and common pedagogical approach for teaching algebra and algebraic reasoning. In this approach, a bag (Sawyer, 1943), a cloud (Mason, Graham & Johnston-Wilder, 2005) or something similar is used to denote a number. Among many different uses of this approach in many different situations and for many different purposes, let us choose one that is most related to both introductory algebra and early deductive reasoning: its use in number tricks.

Different authors have treated these tricks differently. Sawyer (1943) suggests, "you think of a number—any number. We will think of this as marbles placed in a bag" (p. 82). Mason *et al.* (2005) suggest, "beside your chosen number in a second column, write a cloud" (p. 21). Of course, the critical issue is not just a matter of taste; it is the way that we use such "transitional" languages to encourage a student to shift from his or her number, to his or her friend's (friends') number(s), and then, to his or her enemy's (enemies') number(s). Not surprisingly, this is reminiscent of the three stages of convincing proposed by (Mason, Burton & Stacey, 1982): convince yourself, convince a friend, convince an enemy (p. 95). To see how such languages may or may not work, let us conduct a thought experiment, part of which is based on my own experiences.

Suppose I do a number trick with my class. The first step is to ask students to choose a number. The numbers will be of various magnitudes and, depending on the knowledge of my students, of different kinds. It is important to notice that each number is specific for the person who has chosen it. Then I give them the instructions and students do the calculation, line by line. Many of them get *the* number that I have decided on. The next step is for them to see that it is not *a* number, it is *the* number. If I am lucky, there are some students who miscalculated the numbers—if the number trick is not as simplistic as the one presented above for the sake of discussion, I can almost always count on such luck. I will name one of these students, Amir.

Amir's friends, who are already convinced, ask him to disclose his starting number. They do the calculations together, find Amir's mistakes and get *the* result. Consider the degree of specificity of Amir's starting number: at the outset it was certainly specific for him, but less specific for his friends; it was just Amir's starting number. Upon dis-closing the starting number it becomes specific for all. Yet there is no structure in view, although they are now all convinced of Amir's mistake.

What if Amir plays the role of an enemy who does not like to disclose his starting number? "Let us find it by force", some students suggest (this is indeed based on my own experience). This suggestion is a fierce attempt to make the number specific. I will try something more peaceful: think of Amir's number as marbles placed in a bag. What if it is not a whole number? This is a minor problem, compared to the great loss of specificity. The pretend number of the pretend marbles placed in a pretend bag is not Amir's number any-more; it is nobody's number. It is a jump too quick from *the* number to *any* number. Children at a very young age "have difficulty in seeing the real as a special case of the possible" (English, 1991, p. 452). I would not do a number trick with concrete-operational children. However, it seems that there is a bit of that stage with formal-operational students, since what I want is to help them to see the specific as a special case of the general. As implied, the little bag as used here has little respect for this long experienced difficulty. Should I use a little cloud instead?

The use of a little cloud for denoting a number that some-one faraway is thinking of (Mason *et al.*, 2005) has more or less the same problem as the use of a little bag, although it is neutral to the nature of the number thought of. The approach of writing a cloud *beside* the chosen number simply does not work here since it is Amir's number, not mine. Much worse, it is badly reminiscent of the object-like perception of letters—so-called fruit salad algebra and shorthand names (MacGregor & Stacey, 1997). There would be little (if any) algebraic (as opposed to pictorial) difference between a *c*loud, a *c*oconut, or a letter, say *c*, if it does not help students to distinguish between the objects themselves and the number of them, or the varying numbers they stand for (Tirosh, Even & Robinson, 1998). Remember those students who, after seeing a proof of the divisibility of the expression $E = n^3 - n$ by 6, were not at all sure whether or not it is also divisible by 6 for $n = 2357$ (see Fischbein, 1983). It seems that the loss of specificity is at stake here.

Specularity

Let me go back to Amir and ask him to write his number on a little piece of paper. I fold the paper to avoid the disclosure of his number. Now I can do whatever I could do with a little bag or a little cloud. In doing so, I have turned his specific number into what I call a *specular number:* a specular number is *spec*ific for the chooser, but it is treated as a parti*cular*, non-specific number; it is a stepping-stone towards the general. In general, specularity best serves the first type of generalization in which we move from "one" specific object to a "set" containing that object; the set could be restricted naturally (*e.g.*, a bag of marbles) and/or by the perception of the objects involved. The following example of early algebra illustrates this latter point. It also sheds light on the meaning and applicability of specularity itself.

The example is about Arti, a boy that at the time of this "friendly talk" was 6 years old. He knew the numbers from 1 to 10 by their names. More importantly, he was aware of each number in the range 2 to 10 as a combination of various two smaller numbers. He also knew the numbers from 11 to 20, not by their names but as combinations of 10 and another number in the range of 1 to 10. I regarded Arti as a successful product of a master's research study, which aimed to develop young children's number sense [1]. The research was based on Neuman's ideas as described by Marton and Booth (1997):

> The capability of experiencing a number between 1 and 10 as all possible combinations of two parts enables one to handle an unlimited number of situations in which either one of the two parts of the whole is unknown—subtraction and missing addend in the former case and addition in the latter. What is more, larger numbers can now be experienced as combinations of their parts. The numbers between 11 and 20, in the first place, can now be experienced as combinations of numbers between 1 and 20, in particular as combinations of 10 and another number in the range. (p. 61)

At the time of this spontaneous talk, Arti was playing with his toys:

Me: A friend of mine wanted you to solve a problem for her [Arti enjoys solving other people's problems]

Arti: What is that?

Me: She told me a number; oops, I am sorry I have forgotten it. Wait for a second. I remember some-thing: she wanted to take away "five" from it and then add "seven".

Arti: Could you please, please, phone her to find out her number?

Me: I am sorry; I haven't got her phone number. But she just wanted to know whether she gets a bigger or smaller number.

Arti was obviously puzzled by the question. The whole question was meaningless for him. He should take away 5 from a somehow mysterious number, the number that my friend thought of and then add 7 to *it.* Note that here the non-specificity of the "starting number" is being carried to the "middle number" and then to the "final number". As a result, it was hard for Arti to comprehend which number is taken away from which number and which number is added to which number [2]. Arti needed some specific numbers to rely on but I wanted him to think of a particular, non-specific number. Of course, specularity is in play here.

Arti was playing with Lego blocks kept in a bucket. At that moment, there were some Lego blocks in the bucket and some on the floor. I took out five blocks from the bucket and then put seven blocks in the bucket. I rephrased the question according to the elements of the new situation. The whole problem took on a new light for Arti. Not only could he now apprehend that there are more blocks in the bucket, but he could also argue for the number of extra blocks as well:

Arti: Five [blocks] goes for that [taken out blocks] and two new [blocks] go in.

This is an amazing argument for a child of Arti's age. More-over, in the subsequent questions, he successfully worked with different combinations of taken away and added blocks and later on just taken away and added numbers. But, how did the bucket of Lego blocks facilitate his argument?

The use of "hands-on" material helped him. However, it is now widely recognized that "although concrete materials may be helpful for students to learn algebra, it is not the materials themselves that provide algebraic meaning or understanding" (Foster, 2007, p. 163). Thus, the problem is about the way the use of concrete materials may facilitate algebraic thinking. Here, the bucket worked for Arti in the same way as the folded paper mentioned above. First of all, it brought the level of the problem down from "think of my friend's number" to "think of *this* number". Of course, the number of the blocks in the bucket was not known to him. Rather, he could apprehend the blocks as a *specific whole.* The bucket itself was hiding the number of the blocks inside. However, we could disclose the number whenever we wanted. This made the variable (the number of the blocks in the bucket) specific, even while it was unknown (Radford, 2010b). In doing so, it helped him go beyond *intuited variables*—"something whose presence is only vaguely adverted through particular instances" (Radford, 2010b, p. 76)—to deal with variables in an explicit way. Indeed, both before and after this friendly talk, Arti could produce a similar argument with the help of his fingers and for any specific number representable on his fingers. The bucket helped him move from an *in-action operation* (in Radford's sense)—represented on his fingers—to *explicit* operations with unknown numbers. This is an example of the first type of generalization.

What if the starting number were smaller than the number we were planning to take away? Or what if it were not within the range of numbers known by Arti with their names? Or it were not a natural number at all? Notice how whatever restrictions Arti experiences, with the numbers, they are automatically carried to his thinking of other people's numbers. If there are such restrictions, they will be removed (the second type of generalization). However, such personal restrictions did not inhibit him from doing the first type of generalization. Some of these restrictions are only removed after several years of education. For example, the Lego blocks bucket or "the Pebbles-in-the-Bag" activity (Davis & Maher, 1997) may also be used for developing the concept of negative number. Thus, as a teacher (or educator) it is important for us to realize which type of generalization we are trying to foster.

In the light of specularity

Specularity has not come out of the blue. The notion of reducing the importance of a known number has been noticed in the literature. One notable example is the didactic tactic of "tracking arithmetic". Mason *et al.* (2005) look at the potential of arithmetic as a basis "for a transition from implicit to explicit algebraic thinking" (pp. 60-62). They suggest three recording methods for writing down an operation sequence: function machines, arrows or brackets. Interestingly, in terms of specularity, the last one differs from the first two. In the first two, the task is for students to record an operation sequence without a specific starting number. In the last one, the task is to record an operation sequence with a specific starting number while leaving it intact. For example, consider this sequence: add 5, multi-ply by 3, subtract 1. The learners start with a specific initial number, say 4, and record the sequence as follows:

$$4$$
$$(4) + 5$$
$$((4) + 5) \times 3$$
$$(((4) + 5) \times 3) - 1$$

As Mason *et al.* suggest, a "focus on operations makes the initial 4 much less important [and more specular], since it could change [so it can change]" (p. 61; brackets added).

The next task is to track the starting number through the other calculations (*i.e.*, do other arithmetic but do not touch the starting number). The result is the simplified version of the initial sequence into which the initial number is plugged but not yet calculated: (4) × 3 + 14. The aim is to prompt the learners to see what would happen if the initial number were changed to another number; put simply, to see a specific equality as an identity. The specularity may be enhanced by a slight modification of the task [3]. The class is divided into several groups with each group consisting of three members. Each group works with its own sequence of operations and its own starting number. In each group, one member writes down the expressions using brackets, another member does all the calculations to get the final numerical answer, and the third member tracks the starting number. Now the groups find the "new" final answer resulting from the simplified expression of their own. It is a piece of magic: for each group, the "new" final answer is exactly the same as the "old" one [4]. The importance of each group's calculation sequence has been reduced and a *specific sequence* has been turned into a *specular sequence.* Meanwhile, the importance of each group's initial number has also been reduced: if that initial number works for one group, why should it not work for another? As a result, a *specific equality* has been turned into a *specular equality,* that is, an identity.

As suggested, from the three recording methods mentioned by Mason *et al.,* the use of brackets is better adapted to the specularity. I am not saying that the other two methods are necessarily useless or meaningless. In the preceding example, sooner or later you come to a point where the other recording methods can be useful: for example, when the learners start to see a seemingly specific equality as an identity or when you choose the final numerical answer of a sequence of operations and try to find an initial number [5]. Consider that the tasks and the proposed order of them rely on the kind of generalization involved. The same can be said about the use of some other tasks (approaches). For example, you may decide to get learners to work "with sets of possible values" rather than "particular values" and to "operate on *unknowns*" (represented by letters) (Carraher, Schliemann & Brizuela, 2001, p. 131). Of course it differs from specularizing, in which operating on knowns is respected as the first step, and the aim is to foster the first kind of generalization. Does the order of these approaches matter? It depends on which kind of generalization is required. It is important to keep this in mind when reading (or answering) Mason and Pimm's questions.

Back to the initial questions

> *You teach (though wee learn not) a thing unknown*
> *To our late times, the use of specular stone,*
> *Through which all things within without were shown.*
>
> John Donne, To the Countess of Bedford (1633)

Specular stone was an ancient stone through which light could pass if skilfully cut; otherwise, it was just a stone. Similarly, with a specular example, it is just an example, a specific example for the learner through which non-specificity could pass if skilfully used. Specularity prompts the learner to take a course of actions on a specular example, a specific example for the learner that is begging to be treated as a non-specific, particular example of its kind. If the whole process structures the learner's perception of the example at hand "by means of stressing and ignoring various key features", the example turns into a generic example that is "an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general" (Mason & Pimm, 1984, p. 287).

Specularity and genericity are the duals of each other. They are different sides of the coin of the meta-mathematical notion of generalization. They both start with a certain object. The former focuses on the process acted on the object while the latter focuses on the structural features of the object: "unfortunately it is almost impossible to tell whether someone is stressing and ignoring in the same way as you are" (Mason & Pimm, 1984, p. 287). Fortunately, it is not that difficult to see whether someone is performing (rather than experiencing) in the same way that you are.

Specularity takes its strength from the two weaknesses of students: students' reliance on the specific and students' propensity to calculate. As a result, it best serves the first type of generalization in which the learner moves from "one" specific object to a restricted set containing that object. Now I can read and answer the initial questions within a more elaborated framework:

> How can you expose the genericity of an example to someone who sees only its specificity?

For the same reason that it is not wise to foster a structural conception of a mathematical notion before its operational conception (see, for example, Sfard, 1991), it is also not wise to put generecity before specularity. The learner should see the possibility of performing the same process on various already conceived objects of the same *kind* to disclose the structural properties of the objects acted upon and/or to "proceptually encapsulate" the process used (for the latter, see Gray & Tall, 1994). Genericity starts from the end, from a static object. As a result, apart from stressing and ignoring, and repeating the general statement over and over, there is little way to foster the necessary "act" of perception, of seeing the general in the particular. However, it is only a partial answer to the sub-question of the first question:

> Apart from stressing and ignoring, and repeating the general statement over and over, how can the necessary act of perception, of seeing the general in the particular, be fostered?

As mentioned, there is little other way. However, if we read the question as below, the situation changes:

> Apart from stressing and ignoring, and repeating the general statement over and over, how can the necessary act of perception, of *seeing the particular through the general,* be fostered?

First, note that the new version of the question turns the whole issue on its head. Moreover, this approach seems paradoxical, since it presupposes the general. The general in this case refers to the generality of certain already known general processes (like addition) that can be applied to an already known specific object. However, the specific object is just there to be treated as a non-specific, particular example of its *kind*; that is, it is a specular object whose non-specificity is unique to the individual. This is just a part of our answer to the following question:

> How can you discern the extent of the generality perceived by someone else when looking at a particular example together?

This question has a prerequisite sub-question:

> How can you discern *the extent of the particularity* perceived by someone else when looking at a specific example together?

Specularity may help, since you are not only looking at a specific example together, but you are also seeing a process together. Somehow, you have a common sense of the nature of the objects to which the process may be applied, otherwise you could not engage in any communication with each other. It is hard, if not impossible, to know the restrictions *each one of you* may have put on the objects involved. As a teacher (or educator) limited within your own restrictions, you may noticeably prompt your students to pass from consideration of one specific object to the consideration of one non-specific, particular object to which the process may be applied. This is the first type of generalization in which specularity best works. But the variation of the objects to which the process may be applied is determined by the learner's conception of the objects involved. Thus, in a sense, you are seldom looking at a particular example together. However, you may agree upon the sense of the particularity of the already conceived objects if you are to remove any seen or unseen restrictions (for example, at some point you may say, "okay this holds for natural numbers, now let's see…"). This is the second type of generalization for which a different approach is needed, if you are to foster it.

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Notes

[1] The study was conducted by S. Shabani at the Islamic Azad University of Kerman, under my supervision.

[2] Add to this the possibility of "working memory overload" (see, for example, Fayol, Abdi & Gombert, 1987). In our current research, we use different devices for helping children to keep track of the numbers involved.

[3] They mainly use the same sequence of operations for different starting numbers.

[4] If it were not the case for one of the groups, so much the better: there would be a constructive algebraic discussion ahead (see the way we have treated Amir's mistake above).

[5] Interestingly, Mason *et al.* also use brackets as a starter, and then proceed to the other two methods in the course of "undoing" problems.

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