Moore and Less!

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Abstract: This article is the story of a very non-standard, absolutely student-centered multivariable calculus course. The course advocates the so-called problem method in which the problems used are to bridge between what the learners knows and what they are about to know. The main conceptual feature of the course is a unique conceptual story that runs through the course and links all the concepts together.

Keywords: Multivariable calculus, Moore method, problem method, conceptual story

1. INTRODUCTION

This is the story of a very non-standard multivariable calculus course. I think that is worth hearing since it surprised many of my mathematician colleagues. At first glance, it surprised them since it was a non-lecturing course in which no black (or white) board, and no computer was ever used; yet, it was a multivariable calculus course covering all the standard materials of such a course. Thus, my colleagues' first question was "so, what was the means of communication?" Fortunately, the answer is very simple; that is just handouts. Of course, this is a rather lame reply since the question is actually about the content of the means (i.e., handouts) rather than the shape of the means (e.g., blackboard). On a rather different level, it is also about the organization of the course, the role of the lecturer, and the role of the students. Probably, this opening reminds you of one of those calculus reforms in which lecturing is not a favorite choice to deliver the course. Indeed it is, but not for the reasons you might expect.

2. REFORM OR NOT

Usually I teach calculus classes of about 40 students at my own university. I do lecture more often than not. However, course evaluation forms show that my average score on the item about students' collaboration is 19 out of 20. That is "Excellent". Why is this? I cannot phrase the answer better than Krantz [3, p.915]. Thus let me tell you his answer as my own: "I endeavor to create the illusion in my classroom that the students and I are actually carrying on a dialogue, that we are developing the ideas together. In my own way, I am enabling my students to engage in group work, and to participate in discovery learning." In the autumn of 2008 I could not opt for my favorite strategy for a simple fact: the class size was over 130 students.

In the summer of 2008, I was invited to teach a multivariable calculus course at arguably one of the best Iranian universities. The course was in the autumn of the year and the students were a mixture of majors in engineering and science (mathematics, physics and chemistry). 136 students had enrolled in the course and worse, about half of them had previously failed the course at least once. The following table shows the number of students in three categories: First-timers, Second-timers, and Multi-timers (i.e., students who failed the course more than twice.)

First-timers	Second-timers	Multi-timers	Total
72	47	17	136

The table itself shows a rough estimate of the ability (and perhaps motivation) of the students enrolled in the course. To have a more realistic idea it would be informative to know that one of the students had failed the course five times, and one of the first-timers was a silver medalist of the International Mathematical Olympiad (IMO.)

I had a summer ahead of me to find an answer for the most obvious question one might ask: What kind of lecturing works for such a class? My favorite strategy did not seem to be practical. Who would be involved in a dialogue in a class of 136 students with such diverse knowledge, ability, and motivation? How could a student discover what he or she had been told before? Last but not least, how could I encourage students to attend the class where students' attendance is a big issue even in classes of smaller size? I found a solution that solved the problem of students' attendance to a substantial extent, left intact the idea of discovery learning, and encouraged students to dialogue with each other. The **Moore Method** gave me the initial idea of the solution. Also, it implicitly supported me to put into practice an extremely non-traditional approach where traditions are highly ubiquitous. However, I changed the Method to such a great extent that one can hardly recognize Moore anymore.

3. MOORE OR NOT

The mission of the Moore Method (modified or not) is to **do** mathematics [6]. I have strong sympathy with this mission. However, I could hardly adhere to the Method and all its aspects.

First, I would not train my students to become research mathematician for the obvious reasons that only a few of them were students of mathematics, and many of them (at least half) wished just to pass the course more than anything else.

Second, calculus could not be a full-fledge theorem proving course since there are much more to calculus than proving theorems. However, calculus is one of the best courses for cultivating "plausible arguments" [5] during which new concepts might be borne, developed and crystallized. This means calculus is an "open subject" [1], where students see why, first on a plausible ground, and then by actual proofs. I may hope for the latter since they have already experienced what they have to prove.

Third, calculus could not start with a list of definitions followed by "Exercises on the Definitions" [1]. Students need to learn how to calculate since the course is called calculus! But, calculations are severely limited if only done by reference to definitions. Moreover, students also need to understand "the meanings of the definitions" [1]. But, meanings are also severely limited if only referred to a single definition. Thus, whichever way the course goes we need a set of theorems to advance towards meaningful calculations and calculated meanings. This brings us back to the second point in particular, and to the mission of the Moore Method in general.

The mission of the Moore Method is to do mathematics. But, "to do mathematics" could have many different interpretations. I had to choose one for the course and I chose it to mean "to solve problems" since "the problem method is, I am convinced, the way to teach everything. It teaches technique and understanding" [2, p.852]. However, the problem method has its own problems; the main one is a version of the so-called Meno's paradox [4]: how can students solve a problem about something when they do not know what it is? The answer is deeply rooted in the teaching philosophy that I advocate.

4. The TWO PRINCIPLES OF TEACHING

The principles that I espouse are those of phenomenographical view of teaching: "building a relevance structure" and "employing the architecture of variation" [4]. Adhering to the first principle I "should stage situations for learning in which students meet new abstractions, principles, theories, and explanations through events that create a state of suspense. The events…serve to present a shadowy whole, a partial understanding that demands completion and challenges the learner to accomplish it. The whole needs to be made more distinct, and the parts need to be found and then fitted into place, like a jigsaw puzzle that sits on the table half-finished inviting the passerby to discover more of the picture." [4, p.180] According to the first principle, the problems of the course should be inviting, challenging, and part of an interrelated whole. To do this, each problem should have several aspects, some partially coincident with what the learners have already learned, and some related to what the learners are about to learn. Here is the second principle as a natural component of the first: "There has to be a dimension of variation, which is to say that the variation has to be applied to something that might otherwise be fixed, taken for granted."[4, p.185] To put it simply, the second principle says that each problem might bring about different aspects of the concept(s) involved, though not necessarily at the same time. Thus we can choose to focus on some aspects while the other aspects are in the background; then the latter comes to the fore and the former recedes into the background.

I had a summer ahead of me to design my course around those two principles of teaching. The result was the outline of a "conceptual story" that was completed during the course.

5. THE STORY OF A STORY

To plan the story of the course I strictly followed a general rule: each session should meaningfully arise from the previous sessions and should also meaningfully lead to the next sessions. It was why I called it "story". Here I tell the outline of the first eight sessions of story. Consider that more often than not, the main characters start playing their roles before being named.

Sessions 1 and 2, first handout

Surprisingly (for most of my colleagues and probably for the readers of this story), the first idea was the idea of constrained optimization. The first problem was the following taken from Polya [5, p.122]:

> Given two points and a straight line, all in the same plane, both points on the same side of the line. On the given straight line, find a point from which the segment joining the two given points is seen under the greatest possible angle.

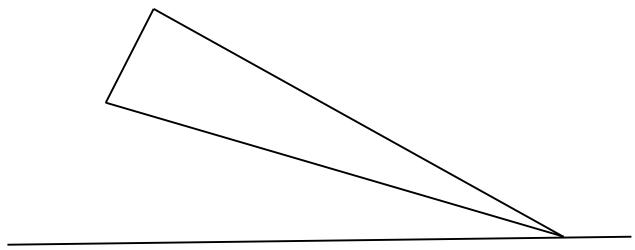

Figure 1. Looking for the best view

On the same handout, students *could read the outline of Polya's own solution to the problem*. The outline demanded completion by which students came to understand and argue for the following figure [5, 124]:

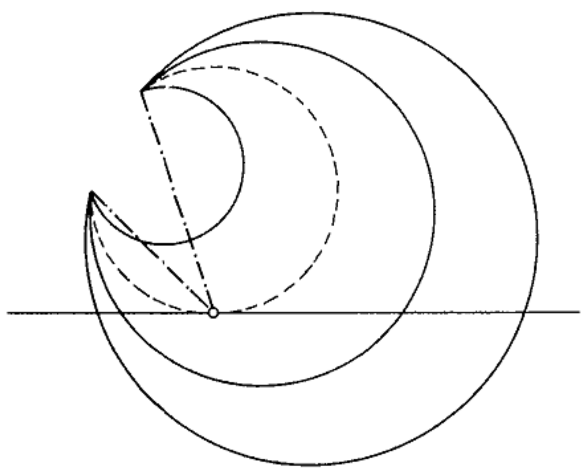

Figure 2. Intersecting circles do not work. The tangent circle does.

For example, they should "argue that intersecting circles do not work since an inner circle that intersects the given line yields a greater angle."

The highlights of Polya's solution for the first problem are:

> Introducing a function of two variables (angle as the function of a variable point in the *plane*); level curves of the function where it remains constant (an arc of circle); and more

importantly, the *pattern* of the tangent level curve: "*the maximum* (or the minimum) *can NOT be attained at a point where the prescribed path crosses a level curve*." [5, p.124]

The next step of the story was varying the context in which students had to apply the same pattern, or, it is better to say, to see it as a pattern! Thus there were some exercises to do so, for example:

On a given straight line find the point which is at the minimum distance from a given point. [5, p.125]

Find the maximum of the area of a triangle inscribed in a given circle and having a given segment AB as its base.

Then the term "level curve" had been used where students had to draw the level curves of some context-free two variable functions alongside the graph of the functions, for example:

$$f(x,y) = x^2 + y^2 \quad , \quad f(x,y) = \sqrt{4 - x^2 - y^2} \quad , \quad f(x,y) = -y^2 \, , \quad f(x,y) = 3y$$

So far, the level curves could be freely spaced by students. Thus the functions above were also to show some similarly looking contour diagrams representing functions of quite different nature. The first session set the scene for the next sessions up to the formal machinery of Lagrange multipliers. Now we were ready to formalize the ideas of the first session step by step. We tackled the idea of *steepness* in the next session.

Session 3, second handout

Here we had a tangled web of different problems embodying different ideas. The first problem was about a mathematical ant on the surface $f(x,y) = x^2 + y^2$. The decisive step was to relate the "actual path" of the ant to the direction it moves. Here is the first problem of the second handout.

Steepness

1. $f(x,y) = x^2 + y^2$.

a) Sketch the surface $z = f(x,y)$ in xyz-space.

b) Sketch the level curves of the function for $z = 0,1,2,3,4$.

c) Imagine a mathematical ant is on the surface at the point where $x = 1, y = 1$. Plot the place of the ant on the surface. Your ant moves on the surface in the direction determined by you on the topographic map. Draw the "actual path" of the ant in each of the following directions from $(1,1)$.

i. In the direction of the vector $\hat{\imath}$,

ii. In the direction of the vector $\hat{\jmath}$,

iii. The direction in which the height of the ant from xy-plane increases fastest (the actual path of the ant upward has the highest slope),

iv. The direction in which the height of the ant from xy-plane decreases fastest (the actual path of the ant downward has the highest slope),

v. The direction in which the height of the ant from xy-plane remains constant.

d) Choose an arbitrary point (x_0, y_0) on one of the level curves of the topographic map. Draw the vectors (iii), (iv), and (v) from (x_0, y_0). Find each vector in terms of x_0 and y_0.

e) Sketch the **vector field** determined by (iii). To do so, choose several points on the level curves of the function and draw the vector (iii).

So far, *the length of the vectors was in the background*. The next two questions brought this feature to the fore:

f) Discuss whether the length of the vectors of the vector field is in agreement with what you are expecting from differences in slope at different points of the surface; if not (I hope so!), re-draw the vector field. Answer the next question after this one!

And finally, based on *the pattern* of the vectors of the vector field, the vector at the point $(0, 0)$ should be found and interpreted.

g) What happens at the point $(0, 0)$? Explain.

As usual, after the first problem of the handout there were some exercises on different aspects of the problem. Some of the exercises were just a repetition of the main problem:

2. $g(x, y) = 4 - x^2 - y^2$. The same questions as Problem 1!

Some of the exercises added new aspects:

3. $z = 2 + \frac{1}{2}x - y$.

a) Your mathematical ant is on the plane at the point where $x = 2, y = 1$. Determine the slope of the path of your ant in each of the following directions.

$$\hat{\imath} \quad , \quad \hat{\jmath} \quad , \quad \hat{\imath} + 2\hat{\jmath}$$

b) Do you agree that “a plane has a constant slope”? Discuss.

c) Sketch the level curves of the function. On the same plane, sketch the vector field representing the direction of steepest ascent at each point.

4. $f(x,y) = x^2 - y^2$. Sketch the level curves. *Geometrically* determine the direction of the fastest increase of the function. Find it in terms of x and y. You are advised to find a vector that is tangent to one of the level curves at an arbitrary point first! If you decide to use my advice, you can also use implicit differentiation!

Now students could *guess* the direction of the fastest increase and justify their guess:

5. $f(x,y) = xy$. Guess the direction of the fastest increase of the function. Justify your guess. You may use my advice!

The biggest step appeared in the last question of session 3 where students needed to find the minimum distance from the origin to a given plane. They could only solve this problem by putting together all the ideas from sessions 1, 2 and 3.

6. Find the minimum distance from the origin to a point on the plane $x + y + z = 1$. You are advised to put together all the ideas you have learned so far! Here is a list:

a) Generalize Polya's idea for finding the minimum distance of a point from a line.
b) Algebraically describe the function that you need to minimize.
c) Give an interpretation of the tangency of two surfaces in terms of the vectors perpendicular to each surface.
d) Find the perpendicular vectors by generalizing the ideas just developed in the current session.

To solve Problem 6, students needed to work with level surfaces. Also notice that through this session the focus was on the direction of the vectors, not their magnitude. Thus, more often than not, the perpendicular vectors were just a multiple of the gradient vectors (that had not been introduced yet). Now we were ready to calculate steepness in the next session.

Session 4, third handout

The first problem of this session was to guide students to calculate a slope for a given surface at a given point and in certain directions.

1. $z = f(x, y) = 9 - x^2 - y^2$.
You start to move on the surface from $(1,2)$ and in the direction $\hat{\imath}$.
a) Show the direction and the path of the move on the figure.
b) We can find a *parametric equation* for the path as follows:

$$\big(x(t), y(t)\big) = (1,2) + t(1,0); \text{ thus}$$
$$x(t) = \cdots,$$
$$y(t) = \cdots, \text{ and}$$
$$z(t) = f(x(t), y(t)).$$

c) On the figure above, interpret $z(t)$. Using $z(t)$, find the rate of change of z when you move in the direction $\hat{\imath}$ from $(1,2)$.

Then students were led to realize that if they want to have a meaningful quantity that is in agreement with what they expect intuitively from the magnitude of slope at different points of the surface, they should take the magnitude of the specified direction into consideration. Meanwhile, they also found the rate of change of the given function in an arbitrary direction from an arbitrary point.

2. Suppose you use the vector $(2,0)$ instead of $(1,0)$ to find a parametric equation for the path above. Thus

$$\big(x(t), y(t)\big) = (1,2) + t(2,0)$$
$$x(t) = \cdots \quad , \quad y(t) = \cdots \quad , \quad \text{and } z(t) = \cdots.$$

   a) First convince yourself this is indeed a parametric equation for the same path!
   b) Calculate $z'(0)$.
   c) Notice that the number you have found is different from the number you have found in Problem 1. What does this difference mean? If you decide to choose one of these numbers as the "slope" of the path at $(1,2,4)$, which one do you choose? Why?
3. Find the rate of change of z when you move in the direction $\hat{d} = a\hat{\imath} + b\hat{\jmath}$ from (x_0, y_0).
4. Choose two points P and Q on the surface, and two directions $\hat{d}_1$ and $\hat{d}_2$ in a way that when you move in the direction $\hat{d}_1$ from P the "slope" of your path is bigger than when you move in the direction $\hat{d}_2$ from Q. Now multiply $\hat{d}_2$ by a number; use the formula found in Problem 3 and find something counter-intuitive.

The moral result: to give meaning to the slope (the number we assign to it) at a given point and in the direction of the vector $\hat{d}$, we have to take the length of $\hat{d}$ constant. I hope you are convinced that the length we need is equal to 1. From now on, whenever we use "the direction of the vector" we mean a unit vector showing the "direction" of the vector!

Then students were advised to write the answer of Problem 3 as a dot product. This form enabled them to argue about what the *fixed vector* of the dot product represents. I had not introduced the gradient vector yet, nor the formal definition of directional derivative! It was to introduce in the next session.

5. You can write your answer to Problem 3 as a dot product:

$$(-2x_0, -2y_0).(a, b)$$

a) Use the properties of dot product to find the direction of the fastest increase of f at the point (x_0, y_0).
b) What is the rate of increase of f in that direction?
c) On the graph of f, show the path you move in that direction.
d) Sketch some of the level curves of the function. On the same plane, sketch a vector field representing the information of (a) and (b) together.

Sessions 5 and 6, fourth handout

So far, we had many geometrical ideas and some calculations done on the given functions. Now we were ready to do the same calculations on a general two variable function in order to draw out some general formulae and to *name* the concepts already experienced in previous sessions, namely, the partial derivative, gradient, and directional derivative in the plane. Thus the first parts of sessions 5 and 6 were to introduce some names and notations, and the second parts were to do some exercises using the new names and notations; mainly these exercises were quite similar to the problems of the previous sessions. In particular, the last question brought students back to sessions 1 and 2, and made them ready for the algebraic form of the Lagrange multipliers that was about to be introduced in the next session. The following is a snapshot of these two sessions.

7. $z = f(x, y)$. We move in direction $\hat{d} = a\hat{\imath} + b\hat{\jmath}$ from (x_0, y_0). Use the same method as Problem 3 of the third handout to find the rate of change of f.

Hint: you come to the following limit:

$$\lim_{t \to 0} \frac{f(x_0+ta, y_0+tb) - f(x_0, y_0)}{t};$$

rewrite it in the following form and continue:

$$\lim_{t \to 0} \frac{f(x_0+ta, y_0+tb) - f(x_0, y_0+tb) + f(x_0, y_0+tb) - f(x_0, y_0)}{t}.$$

8. You can write your answer to Problem 3 as a dot product:

$$\left(f_x(x_0, y_0), f_y(x_0, y_0)\right).(a, b).$$

Use the properties of dot product to find the direction of the fastest increase of f at the point (x_0, y_0).

DEFINITION If $z = f(x, y)$is a function of two variables, the vector $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ is called the *gradient* of f and is denoted by ∇f.

9. Use ∇f to rewrite the formula found in Problem 7(or 8).

Session 7, fifth handout

In this session I just named, and students just formulated what we had experienced and worked with from the very first sessions, i.e. the idea of Lagrange multipliers.

From the first session on, you have thought of and worked with the main idea of this session. What we do now is just to name it officially and to formalize it with the gradient vector!

Goal. Finding extreme values (maximum or minimum) of a two variables function $f(x, y)$ *subject to* a given relation between x and y.

1. Suppose that the equality constraint $g(x, y) = c$ shows the relation between x and y. Use the following figure to argue that if f has an extreme value at (x_0, y_0) then the level curve of f through (x_0, y_0) is tangent to the given constraint at (x_0, y_0).

2. Think of $g(x, y) = c$ as one of the level curves of $g(x, y)$. Read the tangency of the level curves of f and g in the following equation!

$$\nabla f(x_0, y_0) = \lambda \, \nabla g(x_0, y_0)$$

We can use the equation above to find the constrained extreme values of $f(x, y)$. This is called the *method of Lagrange multipliers*.

Then, as usual we had some problems on this old idea but new machinery. The fifth handout brought the first problem cycle of the course to an end. The story proceeded with problem cycles, cycle by cycle to the end of the course. Each cycle was meaningfully linked to the previous cycles; it also planted seeds of new ideas. The next handout, though simple looking, was the link between the first cycle and the rest of the course.

Session 8, sixth handout

So far, students had computed the gradients of several two-variable functions. Now they were faced with the reverse problem of finding a function which has a given vector field as its gradient. This new problem first appeared in the middle of one other problem: the problem of finding a family of curves orthogonal to the level curves of a given function. The plan was to find the vector field of the vectors tangent to the level curves and then find a function with that vector field as its gradient. The level curves of this newly found function would be the family of curves we were looking for. On a first glance, this is just a problem in which some of the previous ideas nicely link to each other. However, considering that the plan works for a function like $f(x, y) = x^2 - y^2$, but does not work for a function like $f(x, y) = x^2 + y^2$, the main question is where the plan fails and why. This opened the door to the second part of the course, i.e. *vector calculus*.

6. THE COURSE STRUCTURE, RULES AND RESULTS

As mentioned above, each session started with administering the handouts. Usually there was no oral explanation since students were to find the link between the materials of the sessions. Indeed, one part of students' assessment was to write a two page essay about the conceptual story of the course.

In each session, the instructor and two of his assistants walked among students to answer their questions and helped them to go through the problems if they needed such help. At the end of the session, each individual student should give his or her written solutions to the instructor, though many of them had worked together during the session. The collected solutions were marked by the instructor and his assistants and were brought back to the students usually one week after the session. A third assistant provided the solutions of the problems of each session and put them on the homepage of the course, usually one week after the session. This gave students the correct answer as well as the intended standard of writing their work. Indeed, this was very important; it was like we wrote several textbooks for the course: some by students, and one by the instructor with a lag of one week. This was a great opportunity for students to make mistakes and to learn from them!

To a great extent, there was no way to follow the course but to participate in the class. Remember this was one of the goals of the course. There were many calculus textbooks out there, but it was hard to follow them since the course has its own unique story. Moreover, 25% of the students' final mark was due to their class-work handed in at the end of each session. Thus they had to participate in the class if they were looking for that 25%! There were also a midterm (20%), a final (50%) and 5% for writing the conceptual story of the course. Only 9 students failed the course; amazingly *none of them was a first-timer*! Students' results aside, students' reaction to the course as a whole is telling. For the first few sessions, they were all puzzled. They did not know what was going on since they expected me to "teach". Indeed, from time to time, one of them raised his or her hand, asking when I will start teaching! After a while, they got used to the idea, and started enjoying it or accepting it. The following excerpt written by one of the students is just an example of many similar comments about students' experience during the course:

> It was very interesting since it seems that we were not learning anything new (that means, nothing new was ever told). But, in each session we were learning something new and we dealt with new concepts, though we weren't aware of it! At the end, suddenly we have realized that we had learned a lot, and it was because the materials of each session were presented by a series of problems that we should find their answers; as a matter of fact, it was like a thread in our own hand in which using what we knew we were answering what we were about to know (it is why that I am telling nothing new was ever taught since it seemed that we knew it all that we could answer the questions, but indeed the answers themselves were the new lessons.) I think this method could be called spontaneous learning; that means we were inclined to learn without noticing. And one feels that he or she has actually learned since he or she has experienced and has found what was about to learn.

The excerpt above could indeed be written by me! It is very surprising since I never discussed with students about the philosophy of the course in a direct manner. The excerpt also bring to the fore the other side of any pedagogy, i.e. learning.

7 THE NATURE OF LEARNING AND FINAL COMMENTS

Any particular view of teaching builds on and reflects a particular view of learning. The kind of teaching that I adhered to is no exception. The two principles of teaching (i.e. building a relevance structure and employing the architecture of variation) underlay the instructional design of the course. The design was to bring about better learning where it is believed that "learning proceeds from a vague undifferentiated whole to a differentiated and integrated structure of ordered parts." [4, p.138] But, the design had two major issues, both somehow interwoven with the design.

The first issue was about the unintended aspects of the problems used. Consider that more often than not students do not see what you want them to see; sometimes they see more, sometimes they see less. Also consider that each problem could bring about different aspects: some intended, some unintended.

Indeed, there is no way to avoid these phenomena. We should just face them knowingly. For example, consider the "ant problem" where an ant moves on the surface $z = x^2 + y^2$ and the problem is to find the direction in which the height of the ant from xy-plane increases fastest. The vectors are perpendicular to the level curves, and they are also radial vectors. But the latter aspect is not what you want your students to see if you are interested in drawing the former aspect out. However, they see it, and if this is the first problem of this kind that they deal with, they *learn* it as such for a while.

This is a well-known phenomenon, and as it was mentioned, there is no way to avoid it in such an instructional design. However, it is important to mention that it is not necessarily undesirable and indeed the instructor could take full advantage of it since it suggests applying a variation to "something that might otherwise be fixed, taken for granted". [4, p.185] Thus it is quite in agreement with the second principle of teaching, and it is also where the experience of the instructor counts.

To the same extent that the first issue of the design was expected, the second was unexpected. Remember that the whole course followed a conceptual story with many different, though interrelated, characters. Also remember that learning is supposed to proceed from a vague undifferentiated whole to a differentiated and integrated structure of ordered parts. But, the entities involved in the story were so interlocked that separating them was a big unforeseen challenge for some students. For example, the concept of double integral was developed alongside Green's Theorem. As a result, there were some students who whenever faced with a double integral were also looking for a closed curve and a vector field!

An important question remains to be answered. What about the Moore Method? At first glance, it seems that it simply gave me the initial idea for the course. However, there is more to it than simply that. The method has students' independent work at its core, whether you focus on proofs or not. The method has significantly less direct teaching than a traditional course; in fact, the teaching is embedded in the order of the problems you choose, whether you focus on proofs or not. Thus, I believe what I did is an adaptation of the Moore Method to a calculus course, whether we call it the Moore Method or not! There were some problems in the wrong place and wrong time. A notable example is Problem 6, the last problem of session 3; it was too hard for most students. Moreover, the story line obviously reflects my own understanding of the concepts involved. Thus, there were certain important ideas that I could not fit into my story. The most notable example was the general form of the chain rule. I hope that my students learn it when they need it. After all, it is what education is all about: teaching students to learn independently.

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