STUDENTS' EXPERIENCE OF EQUIVALENCE RELATIONS A PHENOMENOGRAPHIC APPROACH

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This paper is based on a doctoral study in which we studied 'lay' students' understanding of equivalence relations through individual task-based interviews. We report a conceptual gap between "the everyday functioning of intelligence and mathematics" as to equivalence relations.

INTRODUCTION

It is "an abstraction, a basic mathematical concept, that includes the way species, phonemes, numbers and many other concepts in many parts of life are best thought of…the name of the concept is "equivalence relation"…it is one of the basic building blocks out of which all mathematical thought is constructed." (Halmos, 1982, pp.245-246)

An equivalence relation is "one of the ideas which helps to form a bridge between the everyday functioning of intelligence and mathematics". (Skemp, 1977, p.173)

In this paper we consider *lay* students' understanding of the notion of equivalence relation. In particular, we report one gap (or two!) between "the everyday functioning of intelligence and mathematics". Despite the fact that the tasks(see below) used do not relate to a *formal educational setting*, we also suggest that it will be useful to pay attention to these gaps in our *standard practice of teaching* the notion of equivalence relation, in which, as Skemp says (ibid, p.137), "we start with everyday examples before defining it mathematically".

LITERATURE

Surprisingly, despite the fact that equivalence relation is one of the most fundamental ideas of mathematics, students' conceptions of it have attracted little attention as a research subject. An exception is a series of papers by Chin & Tall (2000, 2001 and 2002) in which they considered the cognitive growth of "equivalence relation" and "partition" at a time when students have been given the definitions and have been expected to operate in an increasingly "theorem-based" manner (ibid, 2000, p.2). However, as a result of working with students already being exposed to the formal treatment of equivalence relations and partitions the focus of the papers inevitably is on the far end of the bridge, i.e. students' understanding and usage of the formal concepts. Thus, in a sense, we furthered their study by investigating the opposite end of the bridge, i.e., *informal* conception of equivalence relations and partitions. In the discussion of the results we will briefly link these studies together.

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METHODOLOGY

The study is based on a detailed phenomenographic analysis of twenty verbatim transcribed audio-taped interviews with students with varied background experience (see also, Asghari, 2004a, 2004b). The participants comprised four middle school students, four high school students, one first year politics students, one first year law student, six first year mathematics students, two second year physics students, one second year computer science student, and one postgraduate student in mathematics. None of them had any formal previous experience neither of equivalence relations nor of the related concepts used to formulate the definition. In a one-to-one phenomenographic interview, each student was introduced to a set of tasks that were designed having the standard definition of equivalence relations in mind (see below). The interviews had a simple structure; the tasks were posed in order, but the timing of the interviews and questions were contingent on students' responses.

Such a varied range of interviewees remind us of a 'pure phenomenography' in which "the concepts under study are mostly phenomena confronted by subjects in everyday life rather than course material in school." as compared to 'developmental phenomenography' in which the concepts under scrutiny are confined to a formal educational setting and the purpose of the study is to help the subjects of the research, or others with the *similar educational background* to learn (Bowden, 2000, p.3). However, in the case of a concept as basic as an equivalence relation, the line between pure and developmental phenomenography fades out.

The Tasks

First, each student was introduced to the definition of a 'visiting law' while they were told that their first task would be giving an example of a visiting law on the prepared grids. (See figure 1.)

> A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.
>
> A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people/
>
> A visiting law must obey two conditions to satisfy the mad dictator:
>
> 1. When you are in a particular city, you are allowed to visit other people in that city.
> 2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The dictator asks different officials to come up with valid visiting laws, which obey both these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid as figure 1.

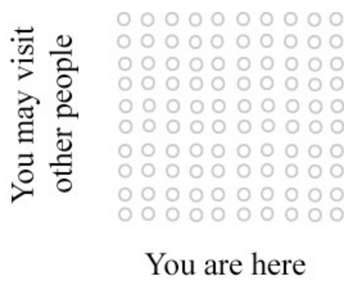

Figure 1: a grid to represent a visiting law

After generating some examples (student-generated, ranging from one example to suggesting a way to generate an example), students were presented with the following three tasks:

Task 1: The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law. Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator deduce the whole of your visiting law.

Task 2: One of the officials, for creating an example, uses other officials' examples: he takes two valid examples and put their common points in his own grid. Is the grid that he makes a valid example? [In the discussion following this is termed *the intersection task*]

Task 3: Another official takes two valid examples and puts all of their points in his own grid. Is the grid that he makes a valid example? [Hereafter, this is termed *the union task*]

Our account of equivalence relations when we designed the tasks

Let us use the eloquent, but still informal, account of equivalence relations given by Skemp (1977).He begins by introducing methods of sorting the elements of a parent set into sub-classes in which every object in the parent set belongs to one, and only one, subset (a partition of the parent set). He (ibid, p.174) considers two sorting methods: first, starting "with some characteristic properties, and form our sub-sets according to this"; and second, starting "with a particular matching procedure, and sort our set by putting all objects which match in this way into the same sub-set". The particularity of this matching procedure is in its "exactness", i.e. having an exact measure for the sameness; a necessity that if it is achieved, the matching procedure is called an equivalence relation. The exactness of the matching procedure also accounts for the transitive property. In addition to the transitive property, an equivalence relation has two further properties, reflexivity and symmetry (see below).

In the problem given to our students, two cities are matched together if their visiting-cities are the same, or two columns are matched together if they have the same status in each row. (For a thorough analysis of the task see Asghari, 2004a).

RESULTS

Analysis of the written transcripts led to a categorisation related to the variation in students' focus of attention in this particular situation. It was possible for the same student to experience different things at different times. The categories are: Matching procedure, Single-group experience and Multiple-group experience.

Matching Procedure Experience

In this category, the focus is on the matching procedure between individual elements; what students experienced and described is in terms of the elements involved, without resort to a group and/or groups of elements. Before giving an example, it is worth saying that somehow or other the defining properties of an equivalence relation determine an exact matching. So do the defining properties of a visiting law.

A matching procedure was exhibited by Ali (first year high school student) when he was generating an example.

Ali: I choose the very first things (points) haphazardly, and then I am going to match the things that have not been matched up yet.

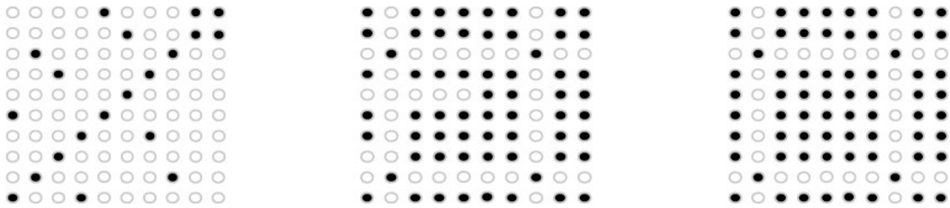

Fig 2: Three stages of Ali's matching procedure

Ali: All right, we start again.

So he paired up city 1 with all the other cities, one-by-one; when two focal columns find something in common, he matched them up, and when they have been already matched or they have nothing in common, he left them as they were. Then he did the same process on city 2 and paired it up and matched it up (if necessary) with all the other city after city 2, and so on. The result of this long process was the middle figure above. Then he continued:

Ali: Now, we are checking from start; it is going to be full (having all points).

And he did so. Eventually the process ended with the right figure above.

Single-Group Experience

In this category, focus is on only one single "group" while all the other elements that do not fall into that group are treated as individuals. The elements in the focal group in one way or another are related to each other while all other elements are in the background as individual elements. Each student in the present study could exemplify this category. However, we have chosen one that at the same time could exemplify different aspects of this category.

Kord (a middle school student) generated the following figures:

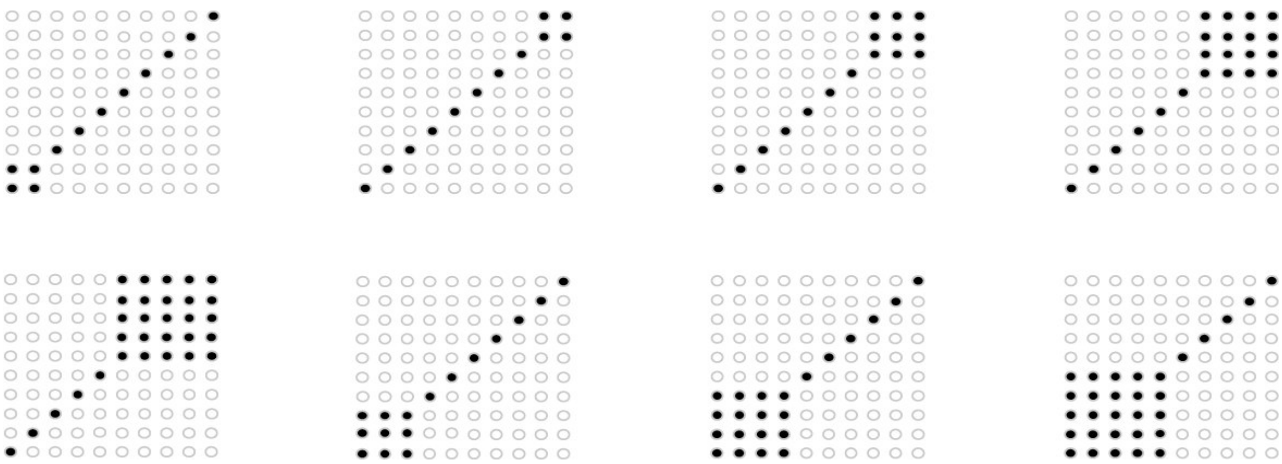

Figure 3: Figures generated by Kord.

Each of these has a square of equivalent points in one corner (lower left or upper right) but in no case did he put together a picture with squares in both corners. Even when confronted with the 'union task', he found it necessary to focus on one square after the other; while he checked whether the square that he has been focusing on has been properly packed, he unpacks the other square and treated its elements on a par with all other individual elements.

Kord: … those that can visit each other are identical and they have no commonality with other cities, so this is correct (this is an example).

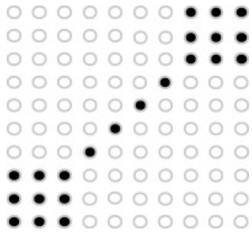

Figure 4: Kord's task putting two examples together.

Since this way of experiencing an equivalence relation has been completely hidden by our formal account (whether formally expressed or informally) we shall give *a few* other examples. Somewhere in his *informal* account of equivalence relation and partition, Skemp[1] (1977, p.174) asks us to imagine that "we are standing on the pavement in London, and in a hurry to get to the station, then we may divide {passing objects} simply into the sub-sets {taxis} and {everything else}". (Let us further his thought experiment) Doing so, we probably could not remember when we went sightseeing in London we divided the very parent set into the sub-sets {double-decker buses designed for tourists} and {everything else}. And still in both situations we do not think of the other passing objects around the world. Given this, it seems in the most practical and/or everyday situation we, ourselves, could exemplify our second category, single-group experience!

Multiple-Group Experience

In this category, "disjoint groups" are experienced; the groups have no elements in common and the elements of each group are related to each other in one way or another. There are only three students who exemplify this category. Let us follow the youngest one (Hess, middle school student) as he dealt with the problem of giving the least amount of information for the following figure on the left, which then was abbreviated to the figure on the right: ("abbreviated" is the way that Hess describes the figure with the least amount of information)

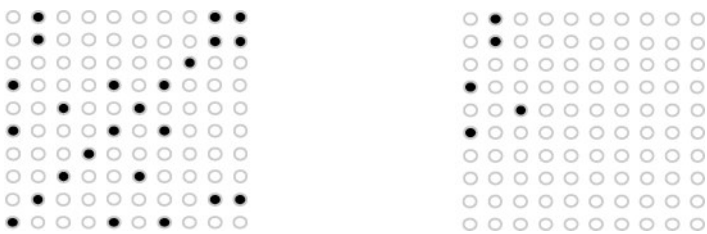

Figure 5: Hess's abbreviation of one of his examples

Hess: For example, one, five and seven make a group (it is the first time that he uses the word "group") with each other, so I only draw five and seven, It doesn't need (to do something) for five and seven, then I see two, nine and ten make a group with each other, I do for two these, it doesn't need for nine and ten; three and six make a group too, four nothing, it make a group for itself, for five, one, no five has been done(suddenly shift to the third category); how many groups are they? It's been finished, eight, it's been finished; that's it.

And then to explain that why this abbreviated figure uniquely determines the original figure he added:

Hess: There is only one case, when we draw the diagonal, the groups are determined; and when the groups were determined there is only one case.

Now, let us enjoy the great extent of the operability of this new idea:

After examining different arguments for the intersection problem he decided to work on the abbreviated figures, since "their abbreviations are themselves" and by using them "our way would be simpler", he suggested.

Hess : Suppose we have an abbreviation, suppose I am deleting certain points, even randomly, it still remain an abbreviation; they have been divided into some groups that have no intersection with each other, certain different groups are created… so if two abbreviations have intersection the intersection is some kind of abbreviation… (In other words) the remained figure is again the abbreviation of another figure.

Reflexivity, Symmetry and Transitivity

Looking at the above categories, we now turn to consider what has happened to the three properties reflexivity, symmetry and transitivity that constitute our normative conception of equivalence relation. In many natural contexts, reflexivity is not made explicit. Family relationships allow A to be a brother of B, but A is not his own brother. Similarly, in some of the earliest formal notions relating to equivalence, the Greek notion of two lines *l, m* being 'parallel' is shown to satisfy the two properties '$a\,\mathsf{P}b$ implies $b\,\mathsf{P}a$' and '$a\,\mathsf{P}c$ and $b\,\mathsf{P}c$ implies $a\,\mathsf{P}b$'. But *a* is not parallel to itself. (How could it be? Two parallel lines have no points in common but *a* has all its points in common with itself). In the case of the example of visiting cities represented on a grid, however, the reflexive law is visible as the main diagonal of the array. (The matter is a little more subtle as the idea of 'matching' usually means matching *two* things. (See Asghari, 2004a for further details.)

Symmetry seems to be the most natural properties of a matching procedure; simply two things are matched together. To see how natural it is, let us recall the example given in matching procedure category where Ali matched up all possible pairs to guarantee examplehood of his figure; however, not quite all possible pairs! Taking symmetry of the matching procedure for granted, he only needed to match forty-five pairs of cities not ninety pairs, as he did so. The ways that our students experienced the geometrical symmetry of each example (see any one of the above examples) or the more algebraic form of symmetry (if (*a, b*) then (*b, a*)) have deeper subtleties.

Our discussion can again start with Skemp who said:

> The importance of the transitive property is that any two elements of the same sub-set in a partition are connected by the equivalence relation. (Skemp, 1977, p. 175)

This suggests that the transitive property is what that makes the vague phrase used in the second (and third) category clear; where we say that "the elements in the focal group in one way or another are related to each other." However, what our students experienced in each single group (of related elements) was the *version of transitivity* formulated above by Euclid and specified by Freudenthal as follows:

> If two objects are equivalent to a third, then they are also mutually equivalent (Freudenthal 1966, p.17).

Let us give an example. Hess is about to explain why the following figure that he has just generated is an example of a visiting law.

Hess: I am going to show that those that have commonality with four are equal to it.

And he did so. And shortly after that, while generalizing his argument he added:

Hess: For each column we check that those that are equal to it, those that must be equal to it, are they equal to it or not.

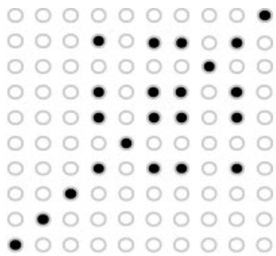

Fig. 6: An example of the visiting law

We will call this version of the property 'F-transitivity' in honour of Freudenthal (following a private communication from Bob Burn). F-transitivity ($a \sim c$ and $b \sim c$ implies $a \sim b$) is equivalent to standard transitivity when dealing with equivalence relations, but it is not satisfied by an *order* relation. The different embodiments of transitivity in order relations and equivalence relations can cause difficulties to students when they are introduced at the same time in a university foundation course (Chin & Tall, 2002).

CONCLUSIONS AND AFTERWORD

Our data suggest that by the standard (and mathematical) treatment of equivalence relation and partition in which we jump from the former to the latter and vice versa, we ignore a gap in everyday experience of the subject, i.e. single-group experience; moreover, If for some purposes we form our focal single-group by a certain matching procedure, it is likely the experience of F-transitivity (not transitivity) that saves us from matching all possible pairs though logically both amount to the same thing.

Being aware of the above deviations from *the standards* could shed some light on our standard practice of teaching equivalence relations and some of its consequences (for example, see the end of the previous section). Furthermore, the above tasks themselves could be used for teaching purposes (though we used them only as a research device).

The first part of The Task of the Mad Dictator (generating an example) was used by a lecturer in one of the top five ranked universities in the UK in a class consisting of fifteen prospective teachers. Following the task he reported:

> The students worked in groups to try to invent new visiting laws. They quickly discovered that just the diagonal and the whole grid were valid laws... one group produced a generic visiting law where each identical equivalence class was coloured the same. They independently 'discovered' the notion of equivalence classes (although they didn't use this terminology of course) and came up with the two main theorems I had on the next seminar's lesson plan.

End note

1- Skemp himself used this example to illustrate that characteristic properties do not necessarily have to have a characteristic property.

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