EXAMPLES, A MISSING LINK

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The purpose of this paper is to draw attention to a missing link regarding the problems involved in 'generating' an example of a defined concept. Through examining students' conceptions while generating (an example), it is argued that these conceptions might be separately linked to a hidden activity: checking (the status of something for being an example). Finally, the missing link between students' conceptions while generating and students' conceptions while checking will be discussed.

INTRODUCTION

Using examples is so blended with our standard practice of teaching mathematics that what is written about the importance of using examples seems to be an expression of triviality. However, to the very same extent that making use of examples seems trivial and mundane, choosing a suitable collection of examples is problematic. It seems that any choice of examples bears an inherent *asymmetric* aspect, i.e. while for the teacher they are examples *of* certain relevant generalisations transferable to other examples to be met, for the students they could remain irrelevant to the target generalization (Mason and Pimm, 1984). When the intended generalization is a concept, usually accompanied by a definition, this divorce of examples from what they exemplify is mainly shown in the literature by examining how students tackle *checking* problems, i.e. checking the status of something for being an example (Tall and Vinner, 1981).

Contrary to the widespread standard teaching practice in which new concepts are introduced by and through teacher-prepared examples accompanied by his or her commentaries on what is worth considering in the prepared examples, there are a few and still experimental non-standard settings in which students are encouraged from the outset to *generate* their own examples. These works mainly arise from the perspective that " mathematics is a constructive activity and is most richly learnt when learners are actively constructing objects, relations, questions, problems and meanings" (Watson and Mason, 2005, p. ix).

The previous two distinct paragraphs, one concerning *checking* and the other concerning *generating*, metaphorically stand for the current view of the literature on these two processes, mainly as two distinct processes (for the examples of this separation see Dahlberg and Housman, 1997, or, Hazzan and Zazkis, 1997). Calling into question this widespread separation between checking and generating is the main theme of the present paper. It will be argued that as far as generating is concerned, its separation from checking lies in the learner's *conception* (of the underlying concepts) and the learner's generating approach rather than the designer's (researcher or teacher) will.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 25-32. Seoul: PME.

BACKGROUND

This paper is based on a wider study aimed at investigating students' understanding of equivalence relations (Asghari and Tall, 2005). The following task (The Mad Dictator Task) was originally designed while having the standard definition of equivalence relations in mind. The task was tried out on twenty students with varied background experience, none of them had any *formal* previous experience of equivalence relations and related concepts. In a one-to-one interview situation each student was introduced to the definition of a 'visiting law' (see below).Then each student was asked to give an example of a visiting law on the prepared grids (see below).

The Mad Dictator Task

A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.

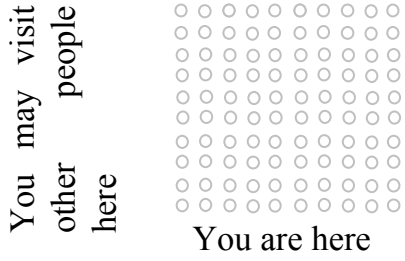
A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.

A visiting law must obey two conditions to satisfy the mad dictator:

1. When you are in a particular city, you are allowed to visit other people in that city.

2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below.



In the previous paper mentioned above, most of the data came from the students' involvement in generating an example. However, in that paper all the tasks involved, including generating, were in the background while students' conceptions of the concepts of interest were in the fore. Now, in this paper, I turn my attention around and start scrutinizing the tasks.

As far as generating and checking are concerned there are a few methodological points worthy of consideration:

First, in our earlier paper mentioned above, according to a methodological choice, the focus of the study was on the outcomes of learning (learned) rather than on the learners. In the present paper, I give more weight to the individuals' voices. However, again it is not the individuals per se that matter; the focus will be on what they reveal about the possible interconnections between generating, checking and the students' conceptions.

Second, in this study generating an example basically means coming up with certain points on the grid where there are incredible blind choices (two to the power of hundred different ways of putting the points on the grid), and where only a tiny portion of all the possible choices constitutes the potential example space (something about one over two to the power of eighty). However, the number of objects (examples) in this tiny portion is still big enough (it is exactly 115975) to surprise the participants by the potential possibility of having an example that is open to them.

Third, in the course of the interview, after generating the first examples each interviewee was asked to generate another one. However, the number of generated examples was mainly determined by the interviewee's will rather than any predetermined plan. As a result, the interviewees' works range from generating only two examples to suggesting a way to generate an example, though they were never asked to explain a general way to generate an example.

Fourth, one of the most subtle points of each interview was to make a decision about checking point, i.e. whether the interviewer should ask the interviewee to check whether his or her self-generated figure is an example or not. This decision was entirely contingent on the interviewee's way of generating his or her example. As a matter of fact, to the same extent that the interviewer could not be aware of all possible conceptions in advance, he could not examine all possible contingencies beforehand. As a general rule, if the interviewer sensed that the way of generating an example reflects a new conception (at least new from the interviewer's point of view) he asked for checking, otherwise that decision was left to the interviewee.

Indeed, the fourth point underlines the missing link stressing in this paper. For a long time, I was not aware of the hidden existence of checking while generating. As a result, generating an example came to a halt as soon as student's generated figure seemed to be an example. However, students' spontaneous attempts to justify their generated figure brought to the fore some complex interrelationships between generating and checking. The next section exemplifies some of these complexities.

Exemplification and Conception

As mentioned before, twenty students with varied background experiences participated in this study. However, in the present paper we only follow two students (Dick and Hess) across two different tasks, namely generating and checking. This is not because these two students and/or their work are typical. They just exemplify how a certain conception may carry different weights in different tasks.

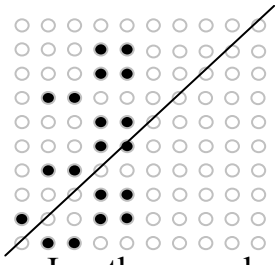
I shall start with Dick (at the time, a first year undergraduate law student) when he was *generating* his first example.

> Dick: this question's been specially designed to confuse.

After a few minutes puzzling over the task and putting some points on the grid, he realized how to represent the first condition:

> Dick: that line (diagonal), that line is the first condition, because you allow to visit people in that city, you see.

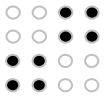
Now, his grid looks like the figure on the right:



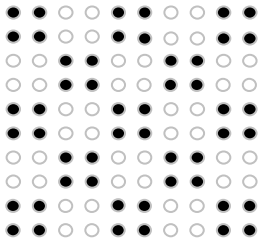
> Dick: so let's *check* this works…um, I think you can't do very complicated *system*, because otherwise a *pair* of cities certainly contradict with each other, I am not sure it works or not, I think I put that wrong, um.

While generating and checking, Dick's focus is on a pair of cities. In other words, his work manifests a *matching conception* in which the focus is on a pair of elements.

Having failed in his first attempt, he turned his attention to a part of the grid, generated the following figure on the left, checked it matching-wise and extended it to the following figure on the right:



> Dick: so *if you do a sort of cross*, I think that satisfy the second condition, um, I also think that would happen, because you get line where you've already filled the centre diagonal which I did before, which means it satisfies the first condition.



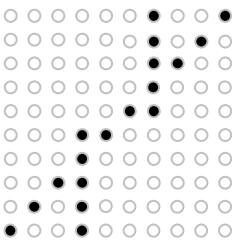
Obviously, his figure satisfies the first condition, but, what about the second condition? Consider that the processes that he made use of was not *predictive,* i.e. making use of it did not guarantee that the product would be an example. As a result, *checking* the status of the product is inevitable. For an informed person, Dick's first example is indeed an example because it consists of two disjoint groups of mutually related elements (here, cities). However, it is not something that Dick *sees* in his figure. Yet, confined himself to matching, He checked it by taking a *random pair*:

> Dick: yah, so that's what I did, so every one of these dots means you can visit that city, … and if you take any *random pair*, I go two and seven, none of them can visit same cities, um, …, two can visit one, but seven can't visit one or two, so that's different, I think that satisfies all conditions.

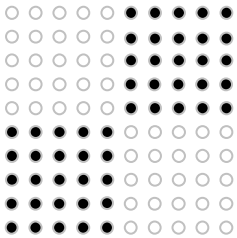
But, his matching conception hindered him to generate another example (at the interviewer's request).

> Dick: um, well, you have to have a *regular pattern* to obey the second condition; because otherwise, um, this is regular, if you, if I want to suppose, I can't hit how it worked out, um, … for every pair, *there's so many pairs*, um, I can't work out, there's so many pairs that we aware if to work out another system, and I can't think another way to do it, um, I stick to this assumption, it has to be a regular system, because otherwise at least one of the pairs conflict, um…I can't think another way…*it is the only example I can think of*.

On the interviewer's insistence, he tries again. Yet, within a matching conception, his focus is mainly on the medium (here, the grid) in which he is about to present another example.



Dick: I just try a *pair*, four and seven having non similar, um; now, I do five and six … they have to cross over with the line in the middle, so I do seven, and that should…go upwards with six… *I think this might also work as a system*.



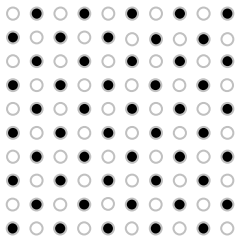
Again, Dick needs to *check* if his system works. This time, for a moment he exhibits a different conception, i.e. *grouping*.

> Dick: um, I just tried a *random example* and it works (ha, ha), um… I just do an example of having the first five cities all be able to visit each other, then the second five cities all be able to visit each other… once again it's a regular system.

However, Dick's grouping experience is not reflected in his next generating attempt.

> Dick: um, once again I *am stuck* for other way to do it, when I did that one (first example); I thought there won't be other ways to do it, now I think of another one which works…for this one (the previous example), I worked from the top to the bottom…, now I am considering filling from left to right to the middle spots…

Soon afterwards he realised that "that would be resulted in the same sort as the second example"! But, still focusing on the figural features, he continued as follows:



> Dick: if you do a system where you fill in blocks of dashes… something possibly diagonal lines, but *I am not sure that system work*, I *must fill in to see it works*…so that each circle, each visiting-city is not next to another visiting-city, um, so that, like that (laughing), … *take a random pair*, say three and six, not same, not same, not same… I think this does work actually (laughing), I just took it and it worked!

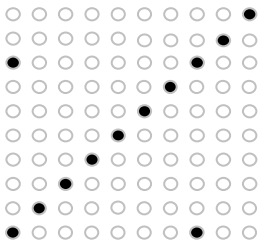
Again it is a vague sense of "a regular system" that leads Dick to his third example. As a result, *checking is inevitable*. When checking, it is only a matching procedure that connects his three examples together. When generating, not only his conception (matching) is hardly noticeable while generating his last example, but also it somehow hindered him to make it!

> Dick: it seems that I shouldn't thought about it (the third example) originally because the second rule is that they must have identical or they mustn't have any in common.

Consider the dual role of the matching conception in generating and checking. It is also worth stressing that it is Dick's approach, rather than matching conception per se, which makes checking inevitable. Hess' experience shows this point.

Hess (at the time, a middle school student) manifested multiple ways of experiencing the situation. After generating the diagonal as his first example, he thought that "it is impossible to have another example" because if "we add another point, they would have a different point, so they never would be identical". Soon afterwards, the same conception (matching)–once hindered him to generate an example–helped him to generate a collection of examples:

Hess: Now, if for every, for example, this point, one, if I put everywhere, for example, I put eight, I must put for eight, one; eight and one are completely the same and others are completely different…so we have infinite cases; no, it is not infinite, but it is a lot…

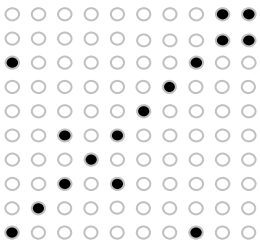


for 3, I put 5 and for 5 I put 3.

Interviewer: and still have you got the previous points?

Hess: it makes no difference…

10, I put 9, also 9, I put 10.



Now, he is looking for something more *general*, certain "property that all of them have":

Hess: `In general, any two` *`symmetric`* `points that we choose we have one (example) about this (the diagonal).`

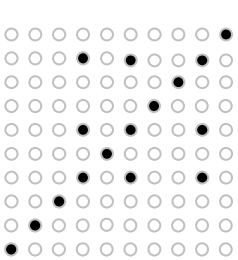
Unlike Dick, Hess envisages what an example might look like. However, like Dick, he only matches together two elements (here, two points), ignoring the possible connections that each one of these two elements might have with the other elements. As a result, some non-examples are counted as examples. However, *in action*, he chose his new pairs somehow keeping them distinct from the old ones. In action or envisaged, in Hess' generating approach, *checking is embedded in from the outset*, yet within a matching conception.

Later on, Hess generated a non-example symmetric figure. To do so, he needed to give weight to an element, and then, its likely relations with more than one element.

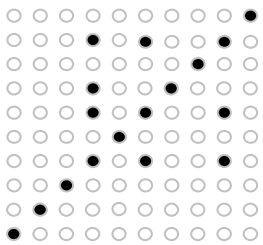
Hess: if 10 has something, that one has 10 too; If 10 has 4, 4 has 10 too, then I must prove that 4 and 10 are equal…if 10 has 5, no, it was rejected... 10 and 4 no longer are equal.

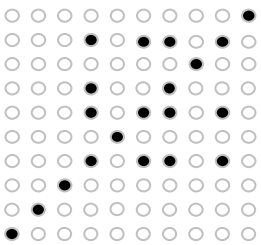
His attempt to provide a non-example symmetric figure plants the seeds of a group-conception reflected in his next generating attempt. Let us enter his work from the middle where he expressed what he has got "as clear as possible and very nicely".

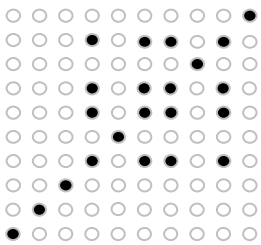
He has already generated the following example on the left:



Hess: My hypothesis is that this one has the conditions of the problem, well, it has the conditions, I suppose I want to make a new condition (a new example) from the previous condition; suppose I choose another point somewhere, I add 4 and 7. (He continues) now, *seven is a member of four.* Then I am working on the seventh column, because seven had itself I determine all the other points but seven; (those are) nine, six and four. (He continues) then in each of the others, I determine seven, because it has just been added.







Hess sees no point to check his newly generated figure. He even does not see his example separated from the process leading to it.

> Interviewer: How you know that this figure is an example?
>
> Hess: I don't want to show it is an example, we suppose it is an example, then I add, and I prove when it is added (the conditions) are satisfied again.
>
> Interviewer: the question is here, you want to give it to the dictator, and say it is the way that people can visit each other; the dictator look at it and say whether you have satisfied the laws. He doesn't want to change it; he saves your paper. You are a dictator; do you accept this one without any change?

Hess' generating attempt manifested a group of pair-wise related elements. To check his figure (on the interviewer's insistence), he turns back to matching for a moment before experiencing a group with a *focal* element related to all the other elements of the group:

> Hess: each two we consider, either they are alike or they are different…Yes. I prove it like this…we investigate for each column, those that are equal to it, those that must be equal to it, are they equal to it, or not?

As it can be seen, one conception is reflected when generating and another conception when checking. Reconciling these conceptions means reconciling generating and checking.

Conclusion

The fragmented experiences described in this paper suggest something in line with what Marton and Booth call *the path of learning*: "that learning proceeds from a vague undifferentiated whole to a differentiated and integrated structure of ordered parts...the more that this principle applies in the individual case, the more successful is the learning that occurs" (Marton and Booth, 1997, p.138).

However, it does not mean that the aspects that have been differentiated and integrated when handling a certain task are also being carried to another task. They are *usage-specific*. In the case of our interest, this means there are certain interconnections between *what* a student conceptualizes and *how* he or she generates an example. In the same vein, there are certain interconnections between *what* he or she conceptualizes and *how* he or she checks the status of something for being an example. This suggests the following schematic figure:

What is experience on the left figure is not necessarily the same as what is experienced on the right figure. They vary in certain aspects; altogether manifest the variation in the students' experiences of the concept involved. Reconciling different aspects of this diversity may result in reconciling generating and checking, and consequently there would be no need for checking after generating. However, this does not mean that there

would be no need for checking activities. Indeed, we need them alongside generating activities, since it is only in the course of tackling different tasks that different aspects of a concept may be differentiated and integrated. It is the way that students may find the missing links between generating and checking.

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