Using Untouchables to Make Arithmetic Structures Touchable

The Case of the Associative Rule

Leyla Khosroshahi & Amir Asghari

Prologue

"*There were 25 lollies on the plate. Mother put 19 more lollies on the plate. The girls ate 17 lollies. How many lollies are there now?*"

The problem above is a typical example of two-step word problems in early years' mathematics textbooks. And, the solution below is a typical solution to such a problem:

$$\begin{array}{ccc} & 1 & \\ & 2 & 5 \\ + & 1 & 9 \\ \hline & 4 & 4 \end{array} \qquad\qquad \begin{array}{ccc} & 3 & 14 \\ & \cancel{4} & \cancel{4} \\ - & 1 & 7 \\ \hline & 2 & 7 \end{array}$$

The solution is step-by-step. The first step is to calculate the number of lollies after the mother added 19 more lollies (25+19 = 44), and the second step is to calculate the number of remaining lollies after the girls ate 17 (44 – 17 = 27). The order of steps somehow is dictated by the wording of the word problem. Written in a line (i.e., 25 + 19 – 17 = 44 – 17 = 27), the steps and the order herein are reminiscent of the so-called left-to-right approach commonly used in early years (Kieran, 1989), somehow working in arithmetic, but an obstacle when moving to algebra (Booth, 1989; Kieran, 1989).

A Different view to Calculations

The left-to-right approach is just the result of one way of *structuring* the original numerical expression. For example, the expression 25 + 19 – 17 is mentally structured as (25 + 19) – 17. This could be due to “a tendency to think about expressions involving binary operations in terms of a sequential procedure” (Larsen, 2010, p.42). Whatever the reason, we share the same view as Booth (1989), stressing the importance of learning activities to assist students’ recognition and use of numerical structure as a prerequisite for understanding algebraic structures. To prepare the ground for such activities let us go back to the expression 25 + 19 – 17.

The expression 25 + 19 – 17 can be seen (structured) in two different ways. One way is to read it as (25 + 19) – 17. Accordingly, there are two consecutive calculations involved, first adding 19 to 25, and second, subtracting 17 from the result of the first calculation. This gives us the final result. However, in earlier years when children are just learning addition and subtraction, the size of the numbers involved could be cumbersome to handle.

There is another way to read the expression 25 + 19 – 17. It can be seen as 25 + (19 – 17). Here again, there are two consecutive calculations involved, first subtracting 17from 19 and second, adding the result (here, 2) to 25. Of course, this gives us the same result as before, but the calculations are less cumbersome to handle.

The following figures show the two ways mentioned.

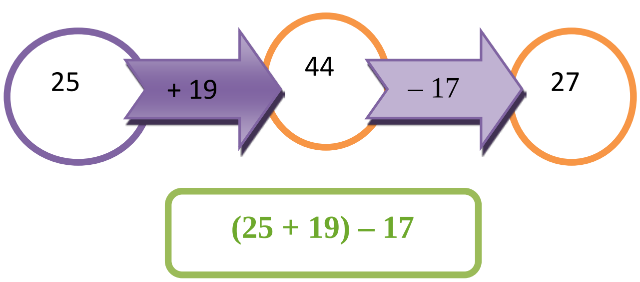

Figure 1. Left-to-right calculation

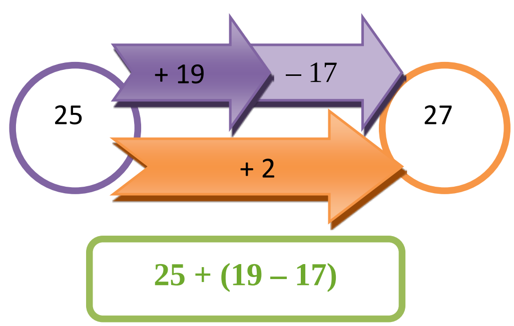

Figure 2. Restructured calculation

Both structures involve two calculations. The second structure not only simplifies the calculations involved, but also opens the way for perceiving the expression as a composition of two changes (adding 19 and then subtracting 17) that yields a third change (adding 2). In doing so, there is a possibility of an early experience of algebra within arithmetic since “one can figure out the total change, which would be valid for any starting point” (Peled & Carraher, 2008, p. 322). In symbols, it means x + 19 – 17 = x + 2. In addition, there is a possibility that the child develops an early “sense of signed numbers as changes” (Peled & Carraher, 2008, p. 322), within arithmetic with whole numbers.

If you feel algebra and signed numbers, even in the sense mentioned above, are too advanced for students in earlier years just read the other examples in Table 1 as an attempt to enable children “to solve simple addition and subtraction problems using a range of efficient mental and written strategies” (ACMNA030, ACARA, 2014).

Expression	Left-to-right calculations	Restructuring the expression
25 + 18 – 19	*(25 + 18) – 19 = 43 – 19 = 24*	*25 + (18 – 19) = 25 – 1 = 24*
37 – 9 + 12	*(37 – 9) + 12 = 28 + 12 = 40*	*37 + (–9 + 12) = 37 + 3 = 40*
76 + 12 + 8	*(76 + 12) + 8 = 88 + 8 = 96*	*76 + (12 + 8) = 76 + 20 = 96*
41 – 7 – 2	*(41 – 7) – 2 = 34 – 2 = 32*	*41 + (–7 – 2) = 41 – 9 = 32*

Note. Children are not expected to be familiar with negative numbers and their calculations. The restructuring used in the rows one, two, and four should be interpreted in terms of a composition of two changes as discussed in the text.

Table 1

Associative Rule

Table 1 provides two methods each for solving expressions a + b + c, a + b – c, a – b + c, and a – b – c, where a, b and c are whole numbers. Using a more advanced language, the arithmetical equality of the two methods for each expression is a direct consequence of the associative rule in which a + b + c = (a + b) + c= a + (b + c), where a, b and c can be any number (positive or negative). This is seemingly well beyond arithmetic of whole numbers. But, it appears in disguise all through primary school mathematics. For example, decomposition that “is a basic component of all subtraction algorithms” (AMSI, 2011, p. 7) implicitly uses the associative property. Students must be able to transform 62 – 5 to (50 + 12) – 5 and see it as 50 + (12 – 5), or alternatively, transform 62 – 5 to 62 – 2– 3 and see it as (62 – 2) – 3. In doing so, they use the two different interpretations of one of the expressions mentioned above. As such, the associative property plays an important though implicit role in students’ ability to change a given problem to an equivalent problem, hence, their ability to use a range of efficient mental and written strategies. It is doubtful that simply telling students about such equality-preserving transformations effectively develops understanding (Carpenter, Franke & Levi, 2003). The underlying principle (i.e., the associative law) should not also be taken as trivial since as the studies of Warren and English (2000) and Warren (2003) show, many students, even at the end of their primary school experience, fail to understand it. These call for Booth’s (1989) suggestion for devising learning activities contributing directly to students’ recognition and use of numerical structures. The next section introduces such activities in the context of the associative law.

Metro Train Activity

The associative law is one way to restructure a numerical sentence like those appearing in two-step word problems, commonly used in the early years. The following activity has been designed for year 2. If you feel that the numbers used are too big for your students or the context is not familiar for them, you can easily change all those providing that you keep the purpose of the activity intact, that is, to assist students to realise different ways of structuring a number sentence and apply the one that is more efficient.

Metro Train Activity

Known-number version. *A metro train left Pars station with 177 passengers. When the train arrived at the next station, Arian station, 68 people boarded the train and 38 people got off.*

- *How many passengers are in the train when it left Arian station?*

Unknown-number version. *A metro train was on its way from Pars station to Arian station. When the train arrived at Arian station, 68 people boarded the train and 38 people got off.*

- *Does the train have more passengers when leaving Arian station than the time it left Pars station?*
- *How many more/less passengers does the train have when leaving Arian station?*
- *If the train had left Pars station with 177 passengers, how many passengers are in the train when it leaves Arian station?*

How This Activity Works

The known-number version somehow invites one to perform as it reads, from left to write: 177 + 68 – 38 = 245 – 38 = 207, where the calculations involved are not easy to be mentally performed.

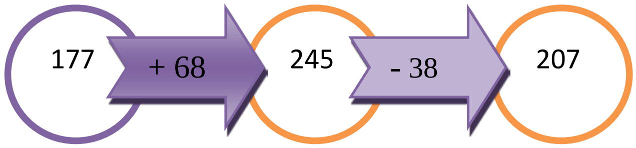

Figure 3

Unlike the known-number version, the unknown-number version initially draws attention away from the final result and puts emphasis on the total change of an unknown quantity that becomes specified later on. In symbolic language, this is seeing x – 38 + 68 as x + (68 - 38) and then, as x + 30 in which x can be specified to be 177 or *any other number*.

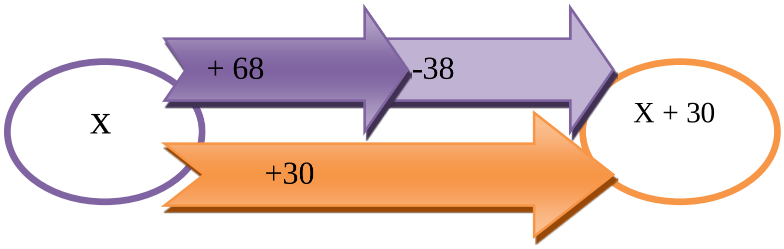

Figure 4

In a language accessible to a year 2, the solution goes as follows where all the middle calculations can be mentally done by standard strategies (for example, adding and subtracting tens).

- *68 is greater than 38, so more people boarded than left. It means that now there are more passengers than before.*
- *68– 38 = 30 so it has 30 more passengers now.*
- *177 + 30 = 207, there are 207 passengers on the train.*

One crucial step is to read the unknown-number version backwards and ask "what if" we knew the initial number of passenger from start. This brings students back to the original sentence 177 + 68 – 38, and allows them to reflect on different ways of structuring the same sentence by "talking with, responding to, and questioning one another as part of the discourse community" (NCTM, 2014, p.30). Accompanied by other similar looking numerical sentences, students have to realise that 177 + 68 – 38 can be restructured as 177 + (68 – 38) where they can easily subtract and add tens, and, 117 + 63 – 8 as (117 + 63) – 8 where they can easily add tens and use the ten complements. All that matter is flexibility to choose an appropriate strategy for solving the problem at hand.

Details matter

If we start with the number of passengers that got off the train and then those who boarded (as it happens in a civilised society), the mental strategy based on a composition of two changes remains intact, but the written strategy needed turns to be a bit more complex. Having come to 177 – 38 + 68, there are two ways to write the combination of the changes involved: (i) to write the sentence as 177 + (68 – 38), or (ii), to write it as 177 + (– 38 + 68). Thus, like many other situations, it is important to find a right order for presenting the tasks to your students. In this regard, a sentence in which both operations are addition is the one that you may want to start with, and the rest (see Table 1) need time to be able to be written by your students (though, they are mentally calculable; see Khosroshahi & Asghari (2013) for some success stories of kindergarten children's recognition and use of structures in the absence of writing symbols).

Finally, it is worth mentioning that just adding the questions regarding the changed number of passengers (the first two parts of unknown-number version) to known-number version hardly ever overcomes students' "compulsion to calculate" (Stacey & MacGregor, 1999) the final answer first, and the final change later.

Epilogue

There is a call for enabling students to use a range of efficient mental and written strategies when solving addition and subtraction problems. For doing so, students should recognize numerical structures and be able to change a problem to an equivalent problem. The purpose of the current article was to suggest an activity to facilitate such understanding in an algebraically fruitful way.

The activity just described above is based on an amazingly simple idea: make a number "untouchable" during calculation. If you are interested in reading more about the theory behind the idea you may consult Asghari (2012). We hope that you make the didactical strategy suggested, one of your own in whatever grade you teach.

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