# BIG BLOCKS OF PROOF

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*In the present study, 193 groups of three students in grades 4 to 6 were assigned a proof-based problem in the field of number theory. The written responses were analysed. Not surprisingly, the analysis showed that the majority of them relied on examples to 'prove' the given statements. However, there was some variation in the ways that examples had been used. Considering the observed variation, 18 students whose proofs were somehow different from each other were invited and interviewed individually for finding more details about their performances. None of them were able to produce accurate formal proofs. However, their performances had an important similarity to mathematicians'.*

## INTRODUCTION

In the last two decades, there has been a growing and widespread consensus on the importance of learning proof among mathematics educators. Concurrently, educational policy makers and curriculum developers has put an emphasis on a certain level of proof in all students' mathematical experiences, even in the elementary schools (Mariotti, 2006, cited in Stylianides, Bieda & Morselli, 2016; National Council of Teachers of Mathematics, 2000, 2014). Proof can be understood as a process of justifying a general statement by any possible mean. As such, the verification by examining a few cases is frequently seen among children and even adults (Reid & Knipping, 2010; Biehler & Kempen, 2013; Lynch & Lockwood, 2017). This is also common in mathematicians (Lynch & Lockwood, 2107), although unlike most students, they are aware of the temporal nature of verification by examples (Weber & Mejia-Ramos, 2011; Mejia-Ramos Fuller, Weber, Rhoads& Samkoff, 2012; Lynch & Lockwood, 2017).

This study addresses the differences and similarities in the students' and mathematicians' uses of examples in the process of proving. Such similarities, if there is any, can be brought to the fore and emphasized in teaching proof. The research questions are:

How do elementary school students use examples in proving? And does there exist any similarities in the ways that students and mathematicians argue?

## LITERATURE REVIEW

A proof is "a connected sequence of assertions for or against a mathematical claim" (Stylianides, 2007, p. 291). Albeit in the most cases, the process of proving does not



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begin by presenting a chain of propositions, but by examining examples, which satisfy or disprove the claim. Research has shown that these examples play a rather different role for mathematicians than the role they play for most students. Mathematicians apply examples for:

1. Attaining intuition about what they want to prove (or refute) (Michener, 1978).
2. Verifying the middle assertions, temporarily, in a long proof for gaining a holistic understanding of the proof (Weber & Mejia-Ramos, 2011; Mejia-Ramos et al., 2012).
3. Finding or seeing the structure of proof, occasionally (Sandefur, Mason, Stylianides & Watson, 2013).

In contrast, the way of using examples by students and even student teachers is ‘simpler’. They apply examples in order to attain the ultimate justification of the given statement (Biehler & Kempen, 2013; Knuth, Choppin & Bieda, 2009). Different names given to this usage of example indicate the extent of its use: ‘empirical arguments’, ‘naïve empiricism’, ‘experimental proofs’, ‘empirical proof scheme’, ‘inductive reasoning’ (Reid & Knipping, 2010).

In general, it seems that the known differences in how and why students and mathematicians use examples in proving is much more than their similarities. The main goal of this study is to find probable similarities.

## METHODOLOGY

The present study adopted a phenomenographic approach (Marton & Booth,1997) to examine the elementary school students’ conceptions of examples in the process of proving. The focus of the study was mainly on the overall variation in the students’ conceptions, rather than the conception of individual students (Asghari, 2007). The research was organized in two phases. In the first phase, written responses of 193 groups of three students were analysed. The students were $4^{th}$ to $6^{th}$ grader, and answered one of following two problems:

> Ali says if anyone gives me three whole numbers, I can add two of them and get an even number. How do you prove Ali’s claim? (64 groups of three students in grade 4 answered to this question).
>
> Ali says if anyone gives me three whole numbers, I can choose two of them such that their sum is divisible by 2. How do you prove Ali's claim? (89 and 40 groups of three students respectively in grades 5 and 6 answered to this question).

A multi-level model was developed based on coding students’ written responses (see Figure 1 in the next section). In the second phase of the study, 18 students who also participated in the first phase were interviewed. The aim was to find different types of conceptions; therefore, they were selected in maximum variation sampling. Samples were from different grades and their responses belonged to different levels of the model. After about two to three months, they individually answered the same problem, also were asked to think loudly about how they thought. Then they were interviewed to explain their responses. The audiotaped semi-structured interviews were transcribed, and then coded according to theoretical coding (Strauss and Corbin, 1998).

## FINDINGS

The developed model does not only show the different roles of examples in the process of proving, but also reveals the student's understanding about the nature of proof (Figure 1).

Then, 18 students were interviewed with the aim of finding more details about the levels of the model and the role of examples in proving. The results confirmed the model and revealed more details about the students' way of thinking as well as similarities between mathematicians and students when they use examples in proving.

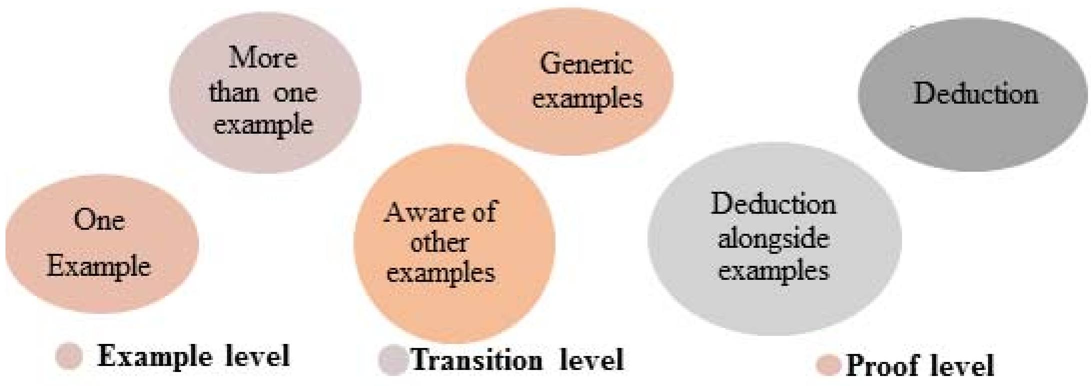


Figure 1: A model for the statues of examples in proving. The figures are schematics.

## THE ROLE OF EXAMPLES IN THE PROVING PROCESS

Figure 1 shows example-based 'proofs' on one side of the spectrum, and deductive proofs on the other side. However, unlike what Figure 1 might imply the two ends are not separated from each other. In fact, as the following descriptions of the levels shows, the levels are our attempt to understand the complexity of students' understanding of the status of examples in proving.

### Example level

In this level, students verify the claim by examining a few cases (We, and in fact, the literature, have already pointed out the spread of this level).

### Transition level

The students' answers are categorized in two groups.

*Aware of other examples:* students, in this level, are aware that the examples they have argued with are not the only possible examples. In a way, they are aware that others might choose some other examples; nevertheless, they are happy with their own choice as a means of proof (and not the means of proof). For instance, a group of 3 students in six grade gave 6 examples and then wrote "there exists other numbers, but we somewhat prove the claim of Ali". These elementary school students had not received any formal teaching in regard to the notion of proof. They also did not have the algebraic experience that could be handy for tackling such unfamiliar problems. Thus, it was predictable that most of the students would not give a fully-fledged mathematical proof. However, it is worth emphasising that these participants realized that their own examples are not the only examples that might be used. From a teacher perspective, such awareness might be considered as one of the first steps towards understanding proofs. Thus, we have called it the transition level to emphasis its importance rather than to claim that these group of students moved, with no return, to something that might be considered as a higher level than the example level.

*Generic examples:* In this category, the other examples are not just any examples; the fall into some classes represented by some other examples. As an example, we have chosen a fifth grader, Helia. She wrote 4 groups of numbers: 8-7-6, 8-12-6, 5-14-17, 13-15-17 and for each group, she showed that there are two numbers whose sums are even. On the surface Helia's answer may be considered in the example level, but interview with her showed that she had categorized numbers in two distinct sets (even and odd numbers); and she had considered 4 possible types of choosing numbers from these two sets: even-even-odd, even-even-even, odd-even-odd, and odd-odd-odd.

> Helia: In these 4 [sets of] cases, I calculated even numbers, odd numbers and also even and odd numbers; and it was divisible by 2.

Then, Helia added that: "if my numbers change, this is still true". It shows that her choices did not depend on any specific numbers, and regarding the evenness or the oddness of her 3 numbers, she could consider them in one of the 4 types of classes. After a few questions and answers, she said: "because the sum of two even [numbers], is an even [number], and the sum of two odd [numbers] is also an even [number]".

Though it might seem repetitive, again we should emphasise that the students who used generic examples did not necessarily move, with no return, to a higher level. (Follow Helia below.)

**Proof level**

At this level, students are able to produce a proof. Nonetheless, examples more or less have the same status with deductive proofs as a means to justify the claims. For instance, the answer of a group of 3 students in grade six was:

> There are 4 types for choosing these 3 numbers. First type: even, even, & even. In this case, Ali chooses 2 evens that give an even. Second type: odd, odd, and odd. Ali obtains an even by choosing 2 odds. Third type: even, even, and odd. Ali obtains an even by choosing 2 evens. Forth type: odd, odd, and even. Ali obtains an even by choosing 2 odds. All even answers are divisible by 2, and in this way the [claim of] problem was proved.

*Deduction alongside example(s):* As Knuth et al. (2009) mention, some secondary school students think examples are substantial for better understanding of their proof.

This was also the case for some of our elementary school students. For example, Atria, a student in the fourth grade, in her interview said: "It is better if there exist an example the meaning of this sentence could be understood better". Some others corresponded a proof to the "final answer" for a problem. For instance, Zahra, a student in the fifth grade, told that she must write some examples along her proof because otherwise "my teacher thinks I cheated, because she knows that if I know the answer, I must write the solution".

*The deduction:* Some students think their proof is adequate and they do not need to give an example. However, it is not necessarily because they think giving examples would be inadequate. For instance, Aria, a student in the fifth grade, thought that it is optional to give or not to give an example along the proof, and when the interviewer asked him to evaluate the answers given by two other students, he gave the complete score to both, one was with an example and the other without, saying that: "because they have proved their claims".

The instances presented above show different ways of example usage in the process of proving. The individual examples given was to clarify the points made, not to distinguish between students. Of course, some of them created better arguments occasionally. But, all of the 18 interviewees, only relied on examples at times. And, more importantly, none fully distinguished the status of examples and proofs as a means of justifying the claims. Yet, surprisingly, some of them showed a kind of performance quite similar to mathematicians.

## SIMILARITIES BETWEEN MATHEMATICIANS' AND STUDENTS' USE OF EXAMPLES IN PROVING

As far as the differences are concerned, the results of this research are in agreement with previous studies: unlike the mathematicians, most students cannot distinguish between the generality obtained by example(s) and the generality obtained by a deductive reasoning (Lynch & Lockwood, 2017). However, as far as the similarities are concerned, one of the findings of these research is quite distinctive.

As we mentioned above, mathematicians sometimes apply examples for temporary justification of middle propositions in the process of a long proof and for a better understanding of the general structure of the proof. In written proofs, these middle propositions are often called *lemma*. The major purpose of them is to reveal the structure of the proof. Mathematicians are well aware that each of them needs a deductive proof on its own. Students do not necessarily have such an awareness. But, it does not stop them to use certain middle propositions which are like **big blocks of proof**, in the process of their reasoning. As an example, let us see another part of the interview with Helia. In her proof, she used the assumption that "the sum of two even numbers is even, and also the sum of two odd numbers is even". When the interviewer asked her to argue for the correctness of her assumption, she verified the first part (the sum of two even numbers is an even number) by examples. Although, Helia used examples for the sum of two even numbers, for demonstrating the truth of the other part (the sum of two odd numbers is an even number) tried to argue deductively:

> Helia: Odd numbers well… if we want to divide them by 2, we obtain a number with a 0.5, and when we get two 0.5, by adding these two numbers, we get a complete number.

In fact, Helia used deductive reasoning as far as possible, and whenever she did not have proper resources for convincing herself or others she used examples. Although Helia could not necessarily distinguish between 'proof' by examples and deductive proof, she completed her proof with the aforementioned proposition, as a big block of the structure of her proof. In the interview, the interviewer allowed Helia to complete her proof with the assumption that the statement is true, instead of pushing her to argue more.

As another example, let us consider Romina, a fifth-grade student. After examining some examples, Romina concluded:

> Romina: Well, certainly, in [three] numbers, if we add two of them, two of them are even or the answer is even. For example, I choose 55, 93, and 1. I add these two [55 and 93] numbers, which are not even … it equals 148. 148 is an even number … it happens when the addition is even or two of our three numbers are even numbers.

When the interviewer questioned her "how do you know the addition of two odd numbers is an even number", she said: "well, we checked it, and we saw that these things I said are true". She, then, was confronted with the question "how do you guarantee your claim is true for other odd numbers". Romina responded that:

> Romina: Because these numbers that I choose are random numbers somewhere between 1 to 100, and we saw that it was true.

Even, the answers placed in the proof level in the first stage of the study were not exempt from examples. For instance, the answer of Matin, Moein, and their teammate (who were in the grade six) seems to be a complete proof, but in the individual interviews with Matin and Moein, both of them, utilized examples to indicate the truth of statements such as "the sum of two even numbers is an even number".

> Matin: I choose 1, 2, and 3. Well, 1 plus 3 is 4. 4 divided by 2 is 2. Now, if the numbers were 1, 3, 5, again 1 plus 3 is 4. 4 divided by 2 is 2. Now, if our three numbers were even, 2, 4, 6, then, 4 plus 2 is 6. 6 is divisible by 2… I understand that the sum of two odd numbers is even, and the sum of two even numbers, is also even.

Helia, Romina, Matin, Moein and some other students showed that they often use such statements in their argument chain, without knowing the exact proof of these statements or even being aware of the necessity of proving them. But, they could use these big blocks of proof in a proper way. For our second problem, these blocks are shown in Figure 2.

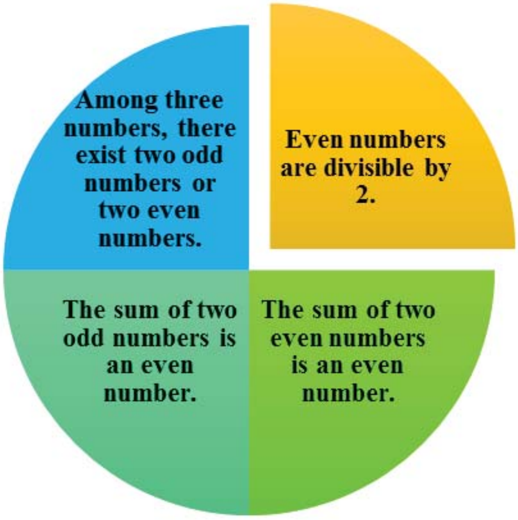


Figure 2: The big blocks of proof for Ali's claim.

## CONCLUSION

The current study confirmed the previous research on the differences between students' and mathematicians' uses of examples in the process of proof. However, our study also revealed an important and potentially useful similarity between the two. Students often apply big blocks (lemmas in the sense of mathematicians) for developing a chain of arguments. This does not necessarily mean that they are able to prove these blocks or are even aware that they need to be proved. Therefore, instead of asking students to prove each of these big blocks from the beginning, it is better to postpone them for an appropriate time. Depending on the type of problem and the student's proving ability, the appropriate time can be right after presenting the proof of the problem at hand, or in the subsequent years, when the proper resources are available. So, it is not necessary to postpone elementary school students' learning of argumentation and proof skills until secondary school. True, they are not able to provide accurate, complete deductive proofs. Yet, by using the proper blocks, a certain level of proof could be accessible to the elementary school students.

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