SYMBOLS IN EARLY ALGEBRA: TO BE OR NOT TO BE?

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As part of a larger project focused on algebraization in pre-school, this paper reports a pre-schooler's engagement in algebraic activities in the absence of algebraic notations. The aim of this paper is to question the necessity of algebraic symbols in early algebraization.

INTRODUCTION

Researches in early algebra are different in their approaches to use symbols. Many studies have focused on the use of symbols, while some more recent studies in the realm of early algebra do not necessitate the use of symbols for elementary students.

Despite much research that has been done in the field of early algebra, very few studies have addressed pre-school's algebraization. Since pre-school children in general are not even able to write number symbols, it is not wise to expect them to use written symbols for variables. Thus, the symbolic algebra or any kind of written symbols in algebra is beyond pre-schoolers' ability.

The current study, focussing on the pre-schoolers' algebraization, shows that symbolization is not a necessary condition for algebraization in pre-school, so algebraic thinking can emerge in algebraic activities and in the absence of algebraic symbols for pre-schoolers. This paper reports a pre-schooler's engagement in algebraic activities in the absence of algebraic notations, or any symbolic notations for that matter. The main question that we want to address here is: How do we design an algebraic situation in the absence of symbols? To answer this question, we exploit a newly introduced idea that takes into account some major obstacles to learning algebra in the early grades.

RELATED LITRATURE

Review of recent research in early algebra shows that there is no common point of view about the use of symbols in early grades among researchers. This is due to differences in both the definition of *algebra* and what the researchers mean by *early*.

Kieren (2004) regards algebraic thinking in the early grades as the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting. In her view of early algebra, algebraization is possible without written symbols. However, Kieren does not give a systematic way for or even any example of algebraization without symbols.

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A research by Carraher (2006) has been guided by the idea that multiple problems and representations for handling unknowns and variables, including algebraic notation itself, can and should become part of children's repertoires as early as possible. His classroom studies suggest that children can handle algebraic concepts and use algebraic notation somewhat earlier than commonly supposed. For example his study shows that given the proper experiences, children as young as eight and nine years of age can learn to comfortably use letters to represent unknown values and can operate on representations involving letters and numbers without having to instantiate them. So there may be no need for algebra education to wait a supposed "transition period" after arithmetic. While Carraher's research subjects are at least at second grade, where children are able to read and write, these findings can't be generalized to pre-schoolers.

Kaput (2008) regards generalization and symbolization as the heart of algebraic reasoning. In his view, the only way a person can make a single statement that applies to multiple instances (i.e., a generalization), without making a repetitive statement about each instance, is to refer to multiple instances through some sort of unifying expression that refers to all of them in some unitary way, in a single statement. But the unifying expression requires some kind of symbolic form, some way to unify the multiplicity. Generalizing is the act of creating that symbolic object. This is where symbolization in the service of generalization- and algebra- starts, both within individuals and historically. So as he states, the use of conventional symbol systems is a necessary condition for an activity to be algebraic but it is certainly not sufficient. Indeed kaput regards a symbolization activity as Algebraic if it involves symbolization in the service of expressing generalizations or in the systematic reasoning with symbolized generalizations using conventional algebraic symbol systems. Moreover, Kaput believes in different levels of algebraic activities. He defines an activity as quasi-algebraic if it satisfies the same conditions except that it may use any symbols, including traditional arithmetic ones, informal ones (including oral speech and physical manipulatives), or idiosyncratic ones. So he qualifies the algebraic use of numbersas quasi-algebraic activity. By algebraic use of numbers he means engaging students in reasoning with numerical statements that are being analyzed not for purposes of computation but for their structure.

A simple comparison between Kieren and Kaput's view of algebraic activities indicates how different understandings of algebraic activities leads to differences in attitudes towards the use of symbols.

While Kaput (2008) regards symbolization as a necessary condition for algebraic activities, Rodford (2011) states that the use of notations is neither a necessary nor a sufficient condition for thinking algebraically. Algebraic thinking is not about using or not using notations but about reasoning in certain ways. What characterizes thinking as algebraic is that it deals with indeterminate quantities conceived of in analytic ways. As he expresses, indeterminacy and analyticity can take several forms. And this is so because algebraic thinking can operate at different layers of generality. Some layers are more concrete, some more general. The most elementary form of algebraic thinking is *factual algebraic thinking*. Indeterminacy appears here in an intuited form: it is expressed through particular instances of the variable in the form of a concrete rule or formula. This embodied form of algebraic thinking can be accessible to most of Grade 2 students. But, results from his study imply that Grade 2 students can also engage in more sophisticated forms of algebraic thinking which has been termed as *contextual algebraic thinking.* In this level, instead of using special known numbers, they use special but unknown numbers without using alphanumeric symbols. So indeterminacy and analyticity will appear in a more explicit way.

Although both Carraher's and Rodford's Researches are on 2nd grade student, they are not unanimous on using symbols. Carraher recommends using symbols in early grades, as early as possible, while Rodford points to other levels of algebraization which are more suitable for early grades.

While Radford introduces an analytical framework in order to distinguish arithmetic and algebra, Asghari (2012) presents the idea of *specularity* that could be used as an operational framemork for algebraization.

THEORETICAL FRAMEWORK

There are major obstacles to overcome to get students to appreciate indeterminacy and analyticity, not least of which is students' reliance on the specific and students' propensity to calculate. The idea of specularity (Asghari, 2012) suggests a systematic way to overcome these obstacles in the generalization situations in which the aim is to get the learner move from "one" specific object to a restricted set containing that object. Indeed, specularity is a direct attempt to use these obstacles as stepping-stones towards the general. Specularity prompts the learner to take *a course of actions on a specific example for the learner* that is begging to be treated as a non-specific, particular example of its kind. Thus, a specular example is just an example whose non-specificity is unique to the individual. Unlike a generic example (in the sense of Mason and Pimm, 1984) that focuses on the structural features of the example at hand, a specular example focuses on the process acted on the example. The process used, supposed to be an already known-for-the-learner general process (like addition) that can be applied to an already known-for-the-learner specific object. However, the specific object is just there to be treated as a non-specific, particular example of its *kind*. In the light of specularity, we may turn a specific-for-the-learner equality into a *specular equality*, that is, an identity. But the variation of the objects (numbers) to which the equality may be applied is determined by the learner's conception of the numbers involved. This would be a major step towards algebra while respecting Rodford (2011)'s indeterminacy and analyticity aspects of algebraic activities at the same time, if skilfully used.

In the light of specularity as a practical framework, our current study tends to show the possibility of early algebraization without using symbols. This paper gives a snapshot of the study.

METHOD

Our current study is a design experiment (Cobb et al., 2003) which is aimed to design algebraic activities for pre-schoolers and investigate how their algebraic thinking and their understanding of variable emerge and improve through those activities. This paper focuses on one pre-schooler to find out the evidences of her algebraic thinking in the absence of written symbols. This case study formed in a pre-school in Iran, in 2012. One of the researchers interviewed Aida—a 6 years old girl—21 times, once a week, while she was doing some tasks designed purposefully in the light of specularity. Aida has a good understanding of numbers, could count objects up to 20 and is able to calculate with numbers less than 5 fluently, sometimes with the aid of her fingers. The whole interviews were video-taped and transcribed for further investigations.

Task 1

In this task there are 4 candies in a bowl on the table and Aida could see the candies. There are also two empty containers on the table. The containers are not transparent, one of them is white and the other one is green. The interviewer shows the inside of the containers to Aida to ensure her that they are empty.

[1] I [Interviewer]: How many candies are there in the bowl?

[2] A [Aida]: four.

[3] I: I want to remove candies from bowl and put them in the containers. Please close your eyes while I do that.

Aida covered her eyes with her hands. The interviewer removes the candies from the bowl and put some of them in the white container and the others in the green one. Aida cannot see how the interviewer distributes the candies among two containers. So she does not know how many candies are there in each container and she cannot see their inside because the researcher has covered them. The interviewer asks her to open her eyes.

[4] I: Do you know how many candies there are in each container?

[5] A: Three in this [pointing to white container] and two, no, one in this [pointing to green one].

[6] I: Three in the white and one in the green? Are you sure?

[7] A: Or if not, one in this [pointing to white container] and three in this [pointing to green one].

[8] I: Ahha... What else may happen?

[9] A: Because they were four, if we take one, this [white] would be 3, this [green] would be one.

[10] I: But I remember none of this happened...

[11] A: If you get one out of three, this [pointing to white] would become two and this [pointing to green one] is two.

Task 2

In this task there are two small dolls, a frog and a ladybird, each one attached to an empty container. In this task interviewer puts the same number of candies in each container. Although Aida knows that the frog and the ladybird have the same number of candies, she doesn't know the exact number of candies. In this task interviewer removes three candies, one by one, from the frog's container and asks her which doll has more candies?

[12] A: Ladybird.

[13] I: How many candies does the ladybird have more than the frog?

[14] A: It [the ladybird] has three more.

[15] I: Now I take one candy from the ladybird...

[16] A: this [ladybird's candies] become a bit less, this [frog's candies] is much less.

[17] I: Which one has more?

[18] A: Again the ladybird.

[19] I: How many candies does the ladybird have more than the frog?

[20] A: Two.

Task 3

In this task, there is an empty container, a bag of hundred candies and more unpacked candies. Interviewer puts some candies in the container while Aida sees the whole process and count the number of candies in the container. Then the interviewer opens the hundred candies bag (while Aida knows that this bag contains hundred candies) and pours it to the container. Next she removes some candies from the container or adds some. Aida sees the process, and interviewer ask her the number of candies added or removed to be sure that she is informed of this number.

In this experiment, the interviewer puts two candies in the container, add the hundred candies bag to it and then remove three candies.

[21] I: Are there more or less than hundred candies in the container?

[22] A: Less than hundred. Because you added two candies and removed three. So it is less than 100. You should add one.

FINDINGS

In this part we are going to discuss the algebraic aspects of Aida's thinking in three preceding tasks. As mentioned before, these three tasks are part of a set of more tasks designed for children to put them in situations where algebraic thinking is possible to emerge. These tasks may help them to make generalizations. Here we explain that how specularity would help them in order to make generalizations, and how this slice of algebraic thinking may happen in the absence of any written symbol.

In task 1 when the interviewer asks Aida to make a partitioning of 4, she says 3 and 1. After Interviewer's insistence on another answer, she announces another answer: 2 and 2. Aida clearly shows that how she makes this new partitioning in line [11]. Removing one from 3 and adding one to 1, gives this new answer. We can translate Aida's saying words to arithmetical writing words as 4 = 3+1= (3-1) + (1+1) = 2+2. So Aida makes this new partitioning by changing the previous one. But how did Aida make the first one? At the first sight we may think that Aida uses number facts, since she has a good understanding of numbers under 5. But deepening in line [9], shows that she uses the same strategy for her first answer. She makes 3 and 1 from 4 and 0, however she never announces 4 and 0 as an answer while she thinks both containers involve candies. So here is the translation of her words to symbols: 4 = 4+0 = (4-1) + (0+1) = 3+1= (3-1) + (1+1) = 2+2. Aida skilfully knows when should calculate the answers and when not. She knows that 3 + 1 = 4 but she changes the structure of the expression in this form: 3 + 1 = (3-1) + (1+1) and then calculates it in some parts: 3+1= (3-1) + (1+1) = 2+2. It is possibly the context of the task that leads her to think in this way. So she uses the same restructuring two times in this task with different starting expressions: the first is 4+0 = (4-1) + (0+1) = 3+1 and the second one is: 3+1= (3-1) + (1+1) = 2+2. As it can be seen, she passes from consideration of the specific-for-her starting numbers (4+0 and 3+1), to new set of numbers while keeping the algebraic relation X+Y = (X-1) + (Y+1) intact. Thus these seemingly arithmetic equalities are more than two specific equalities and they have turned to be *specular equalities*. However, we should be careful about the variation of the numbers to which the equality may be applied. As mentioned before, the variation is determined by the learner's conception of the numbers involved. In the context of this special task, it is possible that for the learner this *general equality* holds just for Xs and Ys with the sum of 4 and not even for all natural sums. However, even with such a small variation in mind, we are intended to think that the task helps Aida to involve in an algebraic activity without using any writing symbols.

Task 2 starts with two equal but unknown quantities: number of frog's candies (say F) and number of ladybird's candies (say L). Aida can identify the doll with more candies after each change in the number of candies. In addition, she can say the difference between these two still unknown numbers. Knowing that F=L, Aida says that L is three more than F-3 (line [14]). In another language, if F = L then L = (F-3) + 3 or L- (F-3) = 3. To put it simpler, while L= F these equalities could be written as: L = (L- 3) + 3 or L- (L-3) =3. Moreover she recognized that L-1 is still 2 more than F-3 or (L-1) = (F-3) + 2 or (L-1) - (L-3) = 2 (line [20]). Although these equalities are symbolically hard for the early algebra level, Aida deals with them fluently without using symbols. The starting number of candies, say L, is an unknown number for Aida, but it is also specific for her, while she can disclose the number of candies whenever she likes and check her answer's correctness. Although it is not evident in the preceding episodes, there were many times in the interview with Aida where she wanted to assure that her answer is right or wrong. The capability of checking the answer is strength of this specular number[1]. The container, involving a special number of candies, plays the role of a specular number for Aida which helps her to pass from a specific-for-her starting number to another number. Again, we should be careful about the variation of numbers to which the above identity may be applied.

The hundred candies bag in task 3 plays the same role as the container for Aida in her algebraizations. Despite the fact that Aida is not capable of doing arithmetic operations with 100, she comprehend 100 as a whole number. When the interviewer asks Aida to compare 100 and (2 +100) -3, she does not calculate 2+100 and then 102-3. She answers the question by restructuring the expression. As she states in line [22], Aida looks at this expression as 100 + (2-3) which is one less than a hundred. So algebraically speaking, she uses two rules of commutativity and associativity in order to answer the question in this way: $(2+100) – 3 = (100 + 2) – 3 = 100 + (2 – 3) = 100 – 1$. It is worth noting that Aida has previous experiences of these two rules, for example in task 1 line [7], Aida make the 1 and 3 partitioning simply from the first 3 and 1 partitioning, which shows her understanding of commutativity. Furthermore, it should be highlighted that Aida's thorough understanding of 100 as a fixed whole helped her to answer the question. Although it is not mentioned here, previous experiments with Aida, when she did not understand 100 as a fixed whole and comprehended it as "very much", shows that she could not give a correct answer to the same question. For example she answered 2+100-3 as 3 less than 100. This answer is due to her incomplete understanding of 100 as "very much" which let her see 2+100 the same as 100, since 2 more than "very much" is still "very much"!

FINAL REMARKS

This study focused on the algebraization of preschoolers while they do not have any experience in reading and writing. The experience of Aida, a pre-schooler engaged in some algebraic activities, shows the potential of specularity as a conceptual tool for making generalizations in the absence of writing symbols. As mentioned before, specularity takes its strength from the two weaknesses of students: their reliance on the specific and their propensity to calculate. So specularity could be better used in earlier grades while the tendency to specific is much more than upper grades. Moreover, the preschoolers' propensity to calculate along with their disabilities in calculations (as the disability of Aida in calculation with 100 in spite of her thorough understanding of 100 as a whole), might be used in favour of specularity.

It is worth repeating that the idea put forward in this paper best works when we try to get the learner to see through the specific-for-the-learner numbers while keeping certain algebraic relations intact. Indeed, a specular number is just a specific number for the learner through which non-specificity could pass if skilfully used. Specularity prompts the learner to take a course of actions on a specular number, a specific number

[1] A computerized version of this task was not as successful as this version, while there was no chance for children to check their answers in reality.

for the learner that is begging to be treated as a non-specific, particular example of its kind. For us, a specific-for-the-learner number is a number that is conceived as a whole. And we are inclined to think that algebra (in the sense of this paper) would be possible with such sense of number. That is why Aida was a good case, and that is why we believe that algebra is possible even within a very small range of numbers, say 1 to 5. It to say that early early-algebra (sic) is possible. This is the possibility that we pursue in our current research.

As a final note we would like to add that the absence of writing makes some difficulties for children. For example, the absence of documentation places a large memory burden on children to keep track of the number involved. So in our main study, this memory burden is to be reduced by the design of activities.

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