Firoozbakht's Conjecture

Ian Stewart
24/06/2022

Prime numbers are among the most important, but also the most puzzling, concepts in the whole of mathematics. Their definition is simple: a whole number is prime if it cannot be obtained by multiplying two smaller whole numbers together. For example, 5 is prime, but 6 = 2 x 3 is not. Prime numbers are basic to many areas of mathematics, both pure and applied. They form the basis of some of the codes used to ensure Internet security. They are puzzling because they do not follow any simple pattern — unlike, say, squares or powers of two.

On the other hand, primes are not merely random. Some patterns can be found. An obvious one is that every prime except 2 is an odd number. Every prime of the form $4k+1$ is a sum of two perfect squares; no prime of the form $4k+3$ is a sum of two perfect squares. There are also useful approximate formulas related to the distribution of primes: roughly how many primes occur in a given range of numbers. In 1792 the great mathematician Carl Friedrich Gauss noticed that the number of primes less than a given number x is approximately $x/\log x$, that is, x divided by its natural logarithm. Proofs that this is correct were finally found in 1896 by Jacques Hadamard and Charles Jean de la Vallée Poussin, using advanced methods from complex analysis. One interpretation is that the size of the nth prime is roughly $n \log n$.

The Iranian mathematician Farideh Firoozbakht is famous for a conjecture about the relation between the sizes of consecutive primes. It states that the nth root of the nth prime forms a strictly decreasing sequence. In symbols, if p_n is the nth prime, then

$$p_{n+1}^{1/(n+1)} < p_n^{1/n}$$

Using a table of the largest gaps between consecutive primes, Firoozbakht discovered that this inequality is true for all n up to 4,444,000,000,000. Later work by the mathematician Alexei Kourbatov has verified it up to 1.84×10^{19}, but to date, no complete proof or disproof has been found.

Many similar conjectures have been proposed. Firoozbakht's Conjecture is one of the strongest, placing closer limits on the distribution of primes than many others. Luan Alberto Ferreira and Hugo Luiz Mariano, surveying results implied by the conjecture, describe it as "the boldest (and reasonable) known statement about prime gaps." They observe that its truth would imply the truth of many other famous conjectures about primes. Using a recent breakthrough on the gaps between consecutive primes by Yitang Zhang, it has also been proved that the sequence of nth roots of the nth prime decreases infinitely many times.

Firoozbakht's mathematical legacy is an intriguing and important contribution to the centuries-long puzzle of the distribution of prime numbers. It poses a fascinating challenge for future mathematicians to solve. If it is true, many long-sought theorems will follow. If it is false, mathematicians will be led to develop new insights into the sizes of gaps between consecutive primes.