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# Experiencing equivalence but organizing order

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**Abstract** The notion of equivalence relation is arguably one of the most fundamental ideas of mathematics. Accordingly, it plays an important role in teaching mathematics at all levels, whether explicitly or implicitly. Our success in introducing this notion for its own sake or as a means to teach other mathematical concepts, however, depends largely on our own conceptions of it. This paper considers various conceptions of equivalence, in history, in mathematics today, and in mathematics education. It reveals critical differences in the notion of equivalence at different points in history and a meaning for equivalence proposed by mathematicians and mathematics educators that is at variance with the ways that learners may think. These differences call into question the most popular view of the subject: that the mathematical notion of equivalence relation is the result of spelling out our experience of equivalence. Moreover, the findings of this study suggest that the standard definition of an equivalence relation is ill-chosen from a pedagogical point of view but well-crafted from a mathematical point of view.

**Keywords** Equivalence relation · Equivalence · Experience · Organizing · Historical variations

## 1 Introduction

The notion of equivalence relation is arguably one of the most fundamental ideas of mathematics. In fact, Halmos (1982) contended that "[it] is one of the basic building blocks out of which all mathematical thought is constructed" (p. 246). It also plays a central role in mathematics education, either simply as a subject to be taught explicitly or as a foundation, disguised or not, for teaching other mathematical concepts. Yet, in whatever way it has been used, it has *recently* manifested itself in a single form: "it is well known and standard" (Halmos, 1982, p. 245); it is a relation that has three properties, namely that it is reflexive, symmetric, and transitive.

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The apparent simplicity of the standard definition of equivalence relation has caused many people to overlook the choices made in defining it. As a result, some mathematicians claim that it includes the way many concepts "in many parts of life are best thought of" (Halmos, 1982, p. 245); some historians retrospectively write it "back into the 'history' of the subject" (Fowler, 1999, p. 371); some mathematics educators assume that it is embodied in certain situations and suggest ways to induce students to discover it as "the common structure" of those situations (Dienes, 1971, 1976); and some cognitive scientists argue that it manifests itself in most of our mundane experience of "balance" (Johnson, 1987, pp. 96–98). And all in all it has been assumed (and sometimes argued) that it has a "definite internal structure" (Johnson, 1987, p. 97) with three properties: reflexivity, symmetry, and transitivity. Yet, many students have considerable difficulty in grasping the standard account of such an omnipresent concept (Chin & Tall, 2001). This may be due to various reasons, among them the way it is taught.

There are several ways to teach the subject. One may start with "everyday examples before defining it mathematically" (Skemp, 1971, p.173) or may advocate Dienes' program (see above) or may introduce a "problem situation" (in the sense of Mariotti & Fischbein, 1997) to get the intended properties functionally to emerge from the solution of a single problem (Asghari, 2005a) or may think of a "genetic decomposition" and then use some "instructional procedures that motivate the mental activities described in [it]" (Hamdan, 2006, p. 127). These approaches seem to be based on this fundamental assumption that *the mathematical notion of equivalence relation is the result of spelling out our experience of equivalence*. A goal of this paper is to substantiate this claim. To do so, I use the "history" of the subject and analyze it from a *variational standpoint* (see Section 2).

After all, whichever approach we choose, it should be kept in mind that students "come to know in a short time [something] that took humanity thousands of years to construct" (Sinclair, 1990, p. 19). If we ignore this history, we have overlooked "the path that was followed by [our] fathers" (Poincaré, 1899, p. 159; translated by Furinghetti & Radford, 2002, p. 638), and we are left with a well-crafted definition deprived of historical conceptions that for a long time had been "operational before somebody hit on the idea to condense them in a comprehensive theory" (Freudenthal, 1983, p. 543); and as a result of this historical deprivation, all the apparently different teaching strategies may converge to just a sentence, that is, "*it will readily be seen that* this relation is symmetric...it is reflexive...it is also transitive" (Dienes, 1971, p. 140; emphasis added). But indeed the history suggests that the first part of this sentence should be replaced by "it will deliberately be chosen that" (see Section 3, Section 4, and Section 5 below)! However, to read the history, we need a "non-standard" framework that describes the notion of equivalence relation "in its relation to the phenomena for which it was created" (Freudenthal, 1983, p. 32); otherwise, we read again the standard properties! The next section is to introduce this framework.

## 2 How to read history

"[Equivalence relations] give a good example of a piece of mathematics that has been popularised only recently but which has already been retrospectively written back into the 'history' of the subject, even back to Euclid..." (Fowler, 1999, p. 371).

Definitions "have been invented to organize the phenomena...from the concrete world as well as from mathematics" (Freudenthal, 1983, p. 32) and the standard definition of equivalence relation is no exception. Indeed, it is *an* advanced means of organizing within "a chain...of definitions" (Mariotti & Fischbein, 1997, p. 225), a "global organization, that is organizing not a system internally but a category of systems by looking from outside" (Freudenthal, 1973, p. 454). From a mathematical point of view, inventing such a global organization is a great achievement. However, it would be a mistake (though very common: see Section 3) if we look at history through such an organization. History abounds with "what can be called the local organizing of a field" (Freudenthal, 1973, p. 151) in which "concepts and relations are always analyzed up to a certain rather arbitrary frontier" within the field. Yet, for certain concepts, say the concept of our interest (i.e., equivalence relation), we cannot simply examine the "history" from an organizational standpoint, whether locally or globally. To organize, one needs to *reflect* on his or her "previously unreflected activity", making it *conscious* and the subject of *reflection* (Freudenthal, 1991, pp. 96–102). But there is no sign of such reflection, say in Euclid's work. Of course, somehow or other, he organized the fields under his study. We may do so as well, though there are occasions when we are not using the same means of organizing as his. This strongly applies to the use of the concept of equivalence relation for reading his work. To avoid this, we need a framework that accounts for *different ways of experiencing* a phenomenon. *Phenomenography* provides such a framework.

Phenomenography is a research specialization that is particularly aimed at identifying "the variation in how the phenomenon in question might be experienced by people with certain background characteristics...in different situations" (Marton & Booth, 1997, p. 128). Here, it means the variation in the ways that some prominent mathematicians of the past have tackled certain situations that from the vantage point of today's mathematics embody the idea of equivalence relation (i.e., *historical variations*). It could also mean the variation in the ways that students, with no previous formal experience with equivalence relations or related concepts, tackle certain situations embodying the idea of equivalence relation. The latter has been reported extensively elsewhere (see Asghari, 2005b); in the present paper, I just use it to investigate the former. This is in line with a fundamental phenomenographical objective, that of "the very identification of the different ways of experiencing a phenomenon" (Marton & Booth, 1997, p. 128), free from situational elements and deprived of the individual voice and "individual qualities" (Marton, 1981, p. 177). Yet, the results may reveal the "built-in taken-for-granted assumptions in the man-made world" (Marton & Booth, 1997, p. 202)—that is an organized world. This in turn may help us to improve our teaching, since "whenever you fail to get someone to understand something, you have taken something for granted that you should not have taken for granted" (Marton & Booth, 1997, p. 202).

Indeed, the historical experiences show so many taken-for-granted assumptions in the standard account of equivalence relation. This is discussed in the following sections. When reading them, it should be borne in mind that I *organize* (in the sense of Freudenthal) the realm of other people's *experiences* (in a phenomenographical sense).

## 3 Historical experiences

As mentioned above, it is a normal practice to read the so-called historical examples in the light of *the* standard account of equivalence relations. For example, consider proposition 12 in Euclid's book X (Heath, 1926, vol. III, p. 34):

Proposition 12: Magnitudes commensurable with the same magnitude are commensurable with one another also.

Regarding this proposition, Fowler (1999, p. 370) wrote as follows: “We have...a straightforward instance of transitivity in X12...and the relation is clearly reflexive and symmetric.”

Fowler is not alone in giving such interpretations. In this case, he himself has simply carried on the tradition of “retrospectively writing back into the ‘history’ of the subject”. Consider Euclid’s first Common Notion (Heath, 1926, vol. I, p. 155): “Things which are equal to the same thing are also equal to one another.”

In David Joyce’s (2008) guide to the first Common Notion we read: “The first Common Notion could be applied to plane figures to say, for instance, that if a triangle equals a rectangle, and the rectangle equals a square, then the triangle also equals the square.”

Compare this with the way that Euclid himself uses the first Common Notion in the proof of Proposition I of Book I (Heath, 1926, vol. I, p. 241): “Each of the straight lines CA, CB is equal to AB. And things which are equal to the same thing are also equal to one another: therefore CA is also equal to CB.”

Neither Euclid’s first Common Notion per se nor the way that he uses it directly conveys the same thing as Joyce’s interpretation (Joyce also immediately shifts to the *standard* account of the subject). If we want to *de-contextualize* Euclid, it seems that a rather different property is more appropriate to use: *F-transitivity.*

### 3.1 F-transitivity

According to Freudenthal (1966, p. 17), an equivalence relation is a relation (say ~) that possesses the following two properties:

1. A ~ A
2. If B ~ A and C ~ A, then C ~ B

Expressed in words, the two laws 1 and 2 will be:

1′. Every object is equivalent to itself (*reflexivity*).
2′. If two objects are equivalent to a third, then they are also mutually equivalent (*transitivity*).

Although Freudenthal himself called the second property “transitivity,” to distinguish it from the standard transitivity (If A ~ B and B ~ C, then A ~ C), let us call it “F-transitivity” (following a private communication from Bob Burn). Having made the distinction between transitivity and F-transitivity, it can be seen that X12 (Proposition 12; see above) and the first Common Notion are straightforward instances of F-transitivity rather than transitivity. As a matter of fact, it is only in Euclid’s proof of Proposition 13 (X13) that we can find a straightforward instance of the transitivity. In the middle of his proof, he says (Heath, 1926, vol. III, p. 36): “For, if B is commensurable with C, while A is also commensurable with B, A is also commensurable with C.”

In practice, up to Proposition 29 (Book I), any time that Euclid makes use of the first Common Notion (e.g., in Propositions 1, 2, 3, 13, 14, 15), he sticks with F-transitivity. It is only in his *proof* of Proposition 30 that Euclid does not obey his formulation of the first Common Notion, although the formulation of Proposition 30 per se still follows the same structure as the first Common Notion (Heath, 1926, vol. I, p.314): “Straight lines parallel to the same straight line are also parallel to one another.”

The first line of the proof, where Euclid specifies three *generic* straight lines, complies with the form of the proposition: "Let each of the straight lines AB, CD be parallel to EF; I say that AB is also parallel to CD."

Then, in the other part of the proof, he makes use of the transitivity:

> The angle GHF is equal to the angle GKD.
>
> But the angle AGK was also proved equal to the angle GHF.
>
> Therefore, the angle AGK is also equal to the angle GKD.

It is interesting that before this proof any use of the first Common Notion is followed by stating it; here, for the first time, Euclid does not state it. More interestingly, in every other place in which, from our point of view, some kind of *equivalence* is concerned, Euclid has a preference for F-transitivity when he *formalizes* a general statement. However, where *operational* use of that statement is concerned, he freely switches from F-transitivity to transitivity and vice versa. Euclid is not alone in doing so.

The interchangeability of transitivity and F-transitivity is so "natural" that it even deceived Freudenthal in the age of modern mathematics. While in the course of defining equivalence relations, he uses the *term* transitivity for what we called F-transitivity; a few pages on (1966, p. 19) when considering order, he uses the same term: "For every three different members a, b, c, of Z it follows from a < b and b < c, that a < c (transitivity of the <-relation)."

Consider that F-transitivity is equivalent to standard transitivity when dealing with equivalence relations, but it is not satisfied by an *order* relation. In addition to F-transitivity, there is also another property distinguishing equivalence relations from order relations: *symmetry*.

## 3.2 Euclid missed symmetry

Let us first come back to Freudenthal's (1966) definition of an equivalence relation. He explicitly mentions the symmetry property, though as a consequence of transitivity (that is indeed F-transitivity!) and reflexivity rather than a defining property. Having transitivity (read F-transitivity!) and reflexivity it also follows:

> If A ~ B then B ~ A (as we see on replacing B by A, A by B and C by B in the Freudenthal's formulation of transitivity: If B ~ A and C ~ A then C ~ B).

In words: "If an object is equivalent to a second object, then the second object is also equivalent to the first (*symmetry*)."

Again, considering Euclid, we are faced with a different conception. Whenever Euclid deals with some kind of *equivalence*, F-transitivity and transitivity amount to the same thing because of the *symmetry* embedded in the situation. However, we shall stress that "Euclid missed symmetry" (David Joyce, op. cit). The symmetry that he missed is the standard account of symmetry as known today. It is a fact that Euclid never used an "if–then" account of symmetry. Generally speaking, he works with a pair of *two* things that are *equivalent* to each other. The symmetry in the *status* of the *two* things involved is maintained by the given definitions: "*Parallel* straight lines are...", "Those magnitudes are said to be *commensurable* which..." and so on. To highlight this point, we shall compare Euclid's treatment with a *modern* axiomatization of geometry. That is Hilbert's (1971, pp. 10–11).

It is only after establishing the *symmetry* of segment congruence that we read: "Due to the symmetry of segment congruence one may use the expression 'Two segments are congruent to each other.'"

And the symmetry of segment congruence, as we may expect, has been *defined* as follows:

If $AB \equiv A'B'$
Then $A'B' \equiv AB$

*It seems what comes first when conceptualizing is last when formalizing.* In doing the latter, Hilbert also depicts a peculiar aspect of the situation.

### 3.3 The number of elements counts

Interestingly, Hilbert also starts with the F-transitivity of segment congruence and moves to the transitivity and then to the symmetry. This is possible by using the property of F-transitivity for only *two* elements.

Hilbert takes as primary the notion of "congruence" (or "equal") between segments. His first axiom of congruence "requires the *possibility of constructing segments*". Then, in the second axiom, he establishes *F-transitivity.*

> III, 2. If a segment $A'B'$ and a segment $A''B''$, are congruent to the same segment AB, then the segment $A'B'$ is also congruent to the segment $A''B''$, or briefly, if two segments are congruent to a third one they are congruent to each other.

The so-called properties of equivalence relations follow from the axioms. However, Hilbert only names two of them, namely, symmetry and transitivity:

> Since congruence or equality is introduced in geometry only through these axioms, it is by no means obvious that *every segment is congruent to itself*. However, this fact follows from the first two axioms on congruence if the segment AB is constructed on a ray so that it is congruent, say, to $A'B'$ and Axiom III, 2 is applied to the congruencies $AB \equiv A'B'$, $AB \equiv A'B'$.
>
> On the basis of this the *symmetry* and the *transitivity* of segment congruence can be established by an application of Axiom III, 2.

Hilbert's argument to deduce the reflexivity of segment congruence (and based on that its symmetry) resembles Freudenthal's when he deduces the symmetry of equivalence relations in general (see above). However, such an argument in which the transitivity or F-transitivity is applied to only *two objects* never occurs in Euclid's elements. Whenever Euclid applies one of these properties, *three different* elements are involved. In this regard, it seems that even Hilbert has a selective memory. Consider the following two definitions of one of *today's* classical example of an equivalence relation, i.e., the relation of being parallel:

> Euclid's Definition 23 (Book I; Heath, 1926, vol. I, p. 154): Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

> Hilbert's Definition of parallels (1971, p. 25): Two lines are said to be parallel if they lie in the same plane and do not intersect.

Both definitions do not allow that a line be parallel to itself. Both give an equal (symmetric) status to both lines. Now compare the following two statements:

> Euclid's Proposition 30 (Book I; Heath, 1926, vol. I, p. 314): Straight lines parallel to the same straight line are also parallel to one another.

> Hilbert (a requirement equivalent to the Axiom of Parallels) (1971, p. 25): If two lines a, b in a plane do not meet a third line c in the same plane then they also do not meet each other.

Now, if we apply the same argument that Hilbert once used for *two* congruent segments, we are faced with what has been ruled out, i.e., the reflexivity of the relation of being parallel: a | | a (a is parallel to a) follows from a | | b, a | | b. Thus, it is evident that when parallels are concerned, Hilbert only can apply *F-transitivity* to *three* different elements. Moreover, neither he nor Euclid allow that *a line be parallel to itself*. In other words, the relation of being parallel is not *reflexive* in either of these treatments. However, as regards the reflexivity of the notions that are more *visibly* related to *equivalence*, Euclid distinctly differs from Hilbert.

### 3.4 The number of applications counts

Generally speaking, Euclid never expresses a phrase like "*a* is equivalent to (read it 'is equal to', 'is parallel to', 'is commensurable with') itself." He employs a rather *alternative* approach. That is, in a sense, applying "one and the same thing twice." In this regard, let me examine those propositions in Book I where there is a *need* for the *reflexivity* of "equality," namely, the propositions that Heath (1926, vol. I, p. 155) has unequivocally decided that Euclid has used, the following Common Notions:

> C. N. 2. If equals be added to equals, the wholes are equal.
> C. N. 3. If equals be subtracted from equals, the remainders are equal.

In I2 (Book I, Proposition 2), we read the following (in which the square bracket is Heath's (1926, vol. I, p. 244) and A and B are two points on DL and DG, respectively):

> DL is equal to DG.

> And in these DA is equal to DB.

> Therefore the remainder AL is equal to the remainder BG. [C. N. 3]

As the link in the square bracket suggests, this is a straightforward application of C. N. 3 in which (two) equals are subtracted from (two) equals.

In I13 (Heath, 1926, vol. I, pp. 275–276), we have:

> The angle CBE is equal to the two angles CBA, ABE.

> Let the angle EBD be added to each.

> Therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD. [C. N. 2]

If we interpret these lines as an application of the second Common Notion, the middle line (let the angle EBD be added to each) implies the reflexivity of equality, i.e., the angle EBD is equal to itself; even in more absurd form, the angle EBD is equal to the angle EBD (though it seems absurd, it is the way that I have been taught Euclidian geometry!). Euclid himself does not use either of these two forms. As mentioned before, in a sense, he applies a "common" thing twice. Even he uses the word "common." However, in this case, for the sake of fluency, Heath (1926, vol. I, p. 276) has decided to remove the word:

> *Let the angle EBD be added to each*, literally, "let the angle EBD be added (so as to be) common,"... "let the common angle be subtracted" as a translation...would be less unsatisfactory, it is true, but, as it is desirable to use corresponding words when translating the two expression, it seems hopeless to attempt to keep the word "common," and I have therefore said "to each" and "from each" simply.

On the one hand, Euclid's insistence on using the latter approach rather than the reflexivity of "equality" and, on the other hand, the very fact that he never appeals to a reflexivity-style-phrase both suggest the difficulty of applying a relation that is basically experienced between *two* objects on only *one* object. Again, Hilbert has a rabbit in his hat.

### 3.5 Just one is enough

Recall Hilbert's first axiom of congruence requires the possibility of constructing a segment congruent to an assigned segment (Hilbert, 1971). Also recall that based on this axiom and F-transitivity he verified that the segment congruence satisfies the standard properties of an equivalence relation. This suggests something intriguing: if it is somehow guaranteed by the context that each object in the universe of discourse is *equivalent* at least to another object in the same universe, then we can define an equivalence relation by stating *only one* property, i.e., the property of F-transitivity. This is intriguing because in most "natural" contexts one has a *direct experience* of a certain kind of equivalence among the objects (*two* or more than two) of the context; to the same extent, there is hardly any need to specify any properties, let alone the standard properties of equivalence relations. Thus, what is the use of the standard properties?

## 4 Nature vs. nurture or equivalence vs. equivalence relation

In looking at a flower, I can mentally isolate the abstract feature of color as such. This act of abstraction would here be primary while the statement that two flowers have the same color "red" would be based on it; whereas in mathematical abstraction it is the equality which is primary, while the feature with regard to which there is equality comes second and is derived from the equality relation (Weyl, 1949, p. 11).

An equivalence relation is a relation that is intended to mathematically recreate different aspects of our experience of equivalence, most prominently, constructing equivalence classes (i.e., the classes of equivalent elements) and, based on that, discarding and at the same time recreating the so-called characteristic property (or common property) by the "principle of abstraction" by which a new entity is *created* (to use the word so dear to Dedekind).

> Peano has defined a process which he calls definition by abstraction, of which, as he shows, frequent use is made in Mathematics. This process is as follows: when there is any relation which is transitive, symmetrical and (within its field) reflexive, then, if this relation holds between u and v, we define a new entity ∅ (u), which is to be identical with ∅ (v). (Russell, 1903, pp. 219–220)

So far, so good! The problem starts when we are pretending that *the* standard properties of equivalence relations are the result of *spelling* out different aspects of our experience of equivalence. But the interchangeability of transitivity and F-transitivity, the "missing" symmetry, and for example Euclid's reluctance to use reflexivity show that the standard properties of equivalence relations are more than a simple indication of people's experience of equivalence. These properties have been *chosen* to *recreate* this experience. Intriguingly, each one of these chosen properties is at odds with our experience of equivalence. Let us go property by property!

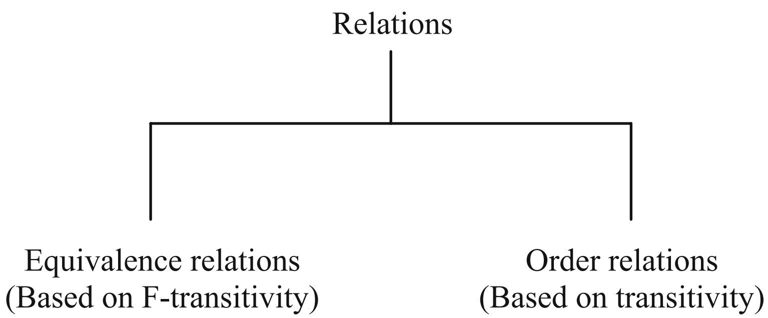
*Transitivity* is one of the defining properties of an order relation: "In a mathematical system the law of transitivity might be at the basis of linear order" (Freudenthal, 1978, pp. 254–255). But, as often as not, "transitivity" also is *chosen* as one of the defining properties of equivalence relations. Tall and Chin (2002, p. 280) point out one of the results of this pedagogically unfortunate choice. They report that some of the students in their study when dealing with the notion of equivalence relation used an embodiment of the transitive law that was compatible with an order relation rather than an equivalence relation:

> In the case of an order relation, it is a natural thought process to imagine the elements ordered in a line, and, in the absence of an embodied image of the notion of equivalence relation, in using the transitive law, it is natural to link to the self-same image.

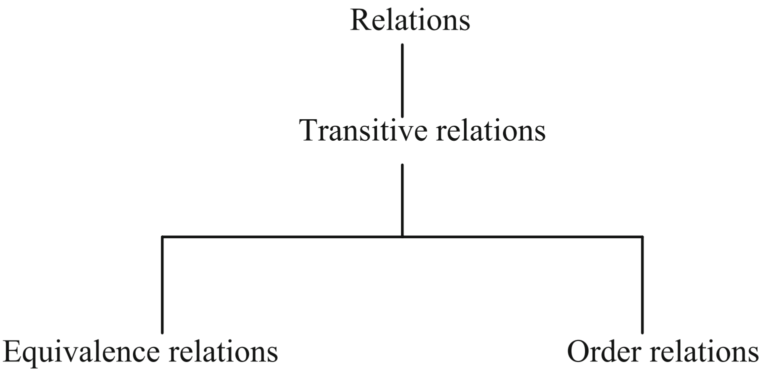
Unlike "transitivity," "F-transitivity" separates equivalence relations from order relations. In the light of the history of the subject, we can see that (see Asghari, 2005b) it could be called "E-transitivity" in honor of Euclid, "G-transitivity" for Gauss (1801), or "C-transitivity" for Cantor (1895). There are many intriguing points about the property of F-transitivity.

First, if it is somehow guaranteed by the context that each object in the universe of discourse is *equivalent* at least to another object in the same universe, then we can define an equivalence relation by stating *only one* property, i.e., the property of F-transitivity. It is what Hilbert did to verify that the segment congruence satisfies *the* standard properties of an equivalence relation.

Second, more importantly, the property of F-transitivity is in harmony with the standard way of defining an *equivalence class* consisting of everything *equivalent* to a *focal* element. Having established the F-transitivity of the given relation, it becomes obvious that any two members of the equivalence class are also equivalent to one another. In a way, the latter is nothing more than repeating the former. Considering that equivalence relations hardly have any other use except for constructing equivalence classes, it seems that giving a definition based on F-transitivity is pedagogically a sound idea. However, mathematically, this gives us a less organized structure (or a local organization). The following figure depicts this structure:



By comparison, the standard definition of equivalence relations (based on transitivity) provides us with a global organization. The following figure depicts this structure:




In the figure above, the two important types of relations, equivalence relations and order relations, *can be* (*logically*) *seen* as particular types of transitive relations. In this regard, the standard definition has a *mathematical* advantage over the definition based on F-transitivity, though the transitive law is more in harmony with the right branch of the figure than the left one.

After all, in a set theoretic setting, F-transitivity and transitivity amount to the same thing, i.e., the *relation* of two things can be drawn from their relation with a third (i.e., F-transitivity!). It is also the case in a natural setting where the *equivalence* of two things can be drawn from their equivalence with a third. However, in the latter case, F-transitivity and transitivity are *indistinguishable* because of a very *natural* property: *symmetry.*

*Symmetry* manifests itself in most of our mundane experience of "balance" (e.g., carrying an equal load in each of our hands—see Johnson, 1987, pp. 96–98) and there is a logic to our experiences of the latter. "[It] has a definite internal structure. This structure has three important properties: symmetry, transitivity, and reflexivity." Seemingly, there is something strange here; we have started with "symmetry" and finished with "symmetry." But consider that the former is the and-account-of-symmetry and the latter is the if–then-account-of-symmetry. In this regard, recall the distinction between Euclid and Hilbert. "Euclid missed symmetry" (i.e., if–then-account-of-symmetry) because the equivalence of the (two) objects was maintained from the outset. For example, in effect, he started by saying that "two segments are congruent to each other" while Hilbert finished by saying the same thing after establishing the symmetry of segment congruence. Generally, this is what we are doing by the standard account of the notion of equivalence relation. It is first and foremost a relation. The latter is defined so that for any two elements (of the underlying set) it is known whether or not *the first is related to the second*. Then, the property of symmetry shows that the initial *order* was redundant and allows us to say that "two objects are equivalent to each other." Thus, the standard account of relations is more in harmony with the right branch of the figures above than the left one (or at least with its root, i.e., equivalence).

Our discussion so far shows it is the standard account of equivalence relations, with its *chosen* properties, that has given rise to Johnson's (and many others inside and outside the mathematics world) "inherent" properties of the balance, not the other way around, as he claims (Johnson, 1987, p. 98): "It is no accident that the properties of the balance schema are just what mathematicians call the 'equivalence relations."

Although Johnson goes too far to claim that the properties of symmetry, transitivity, and reflexivity are *the* properties of "balance," his idea of balance, or roughly speaking, *symmetrical relations* once more highlights the vast variety of situations in which "equivalence" (or in Johnson's term, "balance") is experienced between a pair of *two* things *having* the same status (being in balance). This in turn sheds light on the most troublesome property of an equivalence relation: *reflexivity*.

*Reflexivity* is not experienced directly; "We are, after all, never in a position to balance something with itself" (Johnson, 1987, p. 97). In a more mathematical realm, recall Euclid's reluctance to use a reflexivity-style-phrase whenever some kind of equivalence is concerned. This fact becomes more interesting when we consider that Euclid's definitions are "indications of what is intuitively given" (Weyl, 1949, p. 19). In the case of our interest, it seems that what is intuitively given is that at least *two* objects are involved in our experience of equivalence, and even sometimes the equivalence of these two (or more than two) objects per se is experienced directly. Thus, the property of reflexivity is not simply the result of spelling out our experience. However, it has been *chosen* for certain mathematical reasons, one of which is tightly connected to the fundamental theorem that "allows us to pass at will from an equivalence relation to a partition or back again, by a procedure which, when done twice, leads back to where we started" (Stewart & Tall, 2000, p. 75).

The fundamental theorem skillfully connects the theory of (equivalence) relations to the theory of sets. Here, I point to one connection that I like as my little discovery. That is a little reason for distinguishing between the object *a* and the set {a}. Hausdorff says (1914, p. 12): "A distinction must certainly be made, at least conceptually, between the object a and the set {a} consisting of only this one element (even if the distinction is of no importance from a practical point of view)."

Suppose we have an equivalence relation (by the standard definition). Because of the property of reflexivity, each member is *at least* related to itself. Suppose there is an element being only related to itself, then try to construct its equivalence class. We should accept to have a set consisting of only that one element, or we should consider different cases when stating our theorem. It is well known that mathematicians almost always avoid the latter. What if mathematicians do not take reflexivity into account? Let us see what Ian Stewart says (personal e-mails):

> Ian: Mathematicians could go without reflexivity. Imagine a world in which the textbooks define an equivalence relation by omitting reflexivity. It wouldn't be hard to make everything work. For example, we would redefine the equivalence class [x] to be the set of everything equivalent to x, *together with x itself*. Then all the usual theorems would work. So we could get round the lack of reflexivity by building it into that definition.

> Amir: But it seems that there would be a subtle difficulty. Take "is a sibling of". Assuming that nobody is his or her own brother or sister, it is not a reflexive relation. Now, we find the equivalence classes as you redefined them. We will have many disjoints classes of brothers and sisters, and a lot of singletons, so far so good. Now, we turn back and try to remake the original relation starting from our partitioned world. Will we find the same relation as the one that we started?

> Ian: Don't think that matters. Equivalence relations are almost always used to set up equivalence classes—hence partitions. They seldom have any other use.

This is a *moderate* approach; there is also a *radical* one. Consider that our fundamental theorem is fundamental only in one direction. That is to say, it is important because it verifies that equivalence classes are mutually disjoint. In turn, equivalence relations are important because they can be used to define mutually disjoint equivalence classes. The crux of the problem lies here. In the so-called everyday experiences, there is hardly any need for making the underlying sets explicit. To the same extent, there is hardly any need to take reflexivity into account. But, still, we can split the universe of our discourse into disjoint groups (not sets). Take the family relationship "...is a brother of...." It can easily be seen that we can split people into disjoint groups, each group consisting of the *brothers* (it is plural). Let us see what happens.

You are not your own brother.

There are certain groups of brothers.

Now applying the converse of our Fundamental Theorem (as it is), all of a sudden:

You become your own brother!

It seems that it is only in the mathematics classrooms and mathematics textbooks that "you are your own brother" or "you have the same surname as yourself"! (The latter example has less difficulty when applying the converse of the theorem. However, we can do well without considering the "underlying sets" and the property of "reflexivity").

To think of mathematics without sets is far beyond the scope of this paper. It is also too late to define an equivalence relation without taking reflexivity (and *the* two other properties) into account. However, just being aware of the possible variations makes us wiser and ready to encounter our students' possible difficulties. However, sometimes, these difficulties are just other ways of seeing the situation.

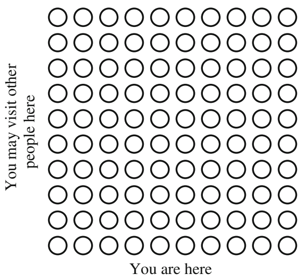
## 5 Yet another way to define equivalence relations

We have already seen that we have (had) many different choices for defining equivalence relations. Yet, there is room for new surprises. Following Freudenthal's view on definitions (see Section 2), I conducted a study on definitions (Asghari, 2004) to see how individuals who had not met the concept of equivalence relation might organize a contextual situation based on the aforementioned concept (as I understood it at the time).

The problem concerned a mad dictator who decreed a "visiting law" to restrict travel between the ten cities in his country. By the decree:

A *visiting-city* of the city, which you are in, is defined as follows: A city where you are allowed to visit other people.

**Fig. 1** The problem grid



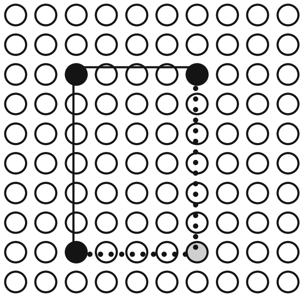


**Fig. 2** The "box property"

And a visiting law must obey two conditions:

1. When you are in a particular city, you are allowed to visit other people in that city.
2. For each pair of cities, either their visiting cities are identical or they must not have any visiting cities in common.

The problem was for the students to come up with valid visiting laws and represent them on a $10 \times 10$ grid (Fig. 1). In this Cartesian situation, one of the students suggested an *unexpected* property that might be called "the box property" (Fig. 2): If three corners of a box (with horizontal and vertical sides) are in the relation, then so is the fourth corner.

The box property provides us with a *new*, *unexpected*, yet *equivalent*, definition of equivalence relation. The following figures (Figs. 3 and 4) show how, having reflexivity and the box property, we can easily deduce symmetry and transitivity:

You may dislike using the box property as one of the defining properties of equivalence relations for several reasons; for example, it just provides us with a local organization, it is far too removed from our experience of equivalence, or it does not look as if there would be any "mathematical relation for which the box concept is a convenient encapsulation" (to answer a question raised by Bob Burn in a personal letter). Yet I hope you agree that it is a *choice*!

## 6 Pedagogical notes

Consider the following question (Tall & Chin, 2002, pp. 275–279):

> Let X = {a, b, c} and the relation ~ be defined where a ~ b, b ~ a, a ~ a, b ~ b, but no other relations hold. Is this an equivalence relation? If not, say why?

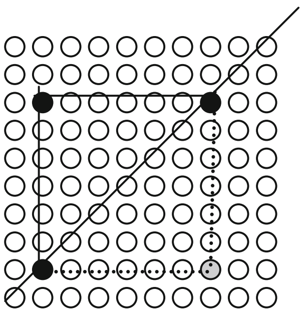


**Fig. 3** Symmetry based on "box" and reflexivity

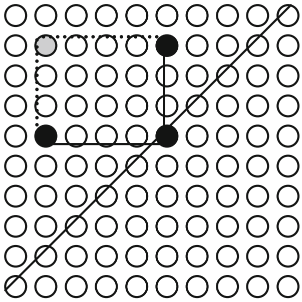


**Fig. 4** Transitivity based on "box" and reflexivity

"It is not, it needs three different elements to make transitivity hold"; this is an "incorrect" reason for a "correct" answer, the like of which everybody who teaches the subject has seen at one time or another. Indeed, as far as handling the standard definition is concerned, this rightly falls into the "incorrect" category. However, the incorrect ways of tackling this particular question appear to stem from a certain deeper conception having some historical counterparts. In this regard, recall that whenever Euclid applies transitivity (or F-transitivity) *three different* elements are involved, and it is so used to draw the equivalence of *two* things from their equivalence with a *third*. Thus, even though the incorrect answer is not based on a "correct" application of the standard formulation, arguably it is not in conflict with our (our students, Euclid and so on) "operational knowledge of transitivity" (Freudenthal, 1983, p. 12). This is also the story of symmetry.

Coming back to the question, when asked, "If the two relations a ~ b, b ~ a are removed from this question, what will happen?" the reply is "symmetry will not hold either" (op. cit., p. 279). In effect, it means it needs two elements to make symmetry hold. More interestingly, when there are two elements, the symmetry property is not dealt with as we formulate it, namely, in the form of an implication. But the common form which the symmetry property is dealt with is the "and" form: a ~ b and b ~a hence symmetric.

All of these remind us of Euclid and Gauss, with a big difference that they did not get low marks for that!

And finally, *reflexivity*, the property that even Russell (1919) was reluctant to accept. Imagine what will happen if we add the relation c ~ c to the ones already defined, a ~ b, b ~ a, a ~ a, b ~ b. It seems that all of a sudden two hitherto different and inequivalent elements, *a* and *b*, become equivalent. It shows to what extent the problem could be at odds with the students' previous experience of equivalence. If they had been taught that, in effect, "the word 'equi-valent' suggests the meaning 'worth the same'" (Skemp, 1971, p. 173), imagine their reactions to the sudden change in the status of two *different* elements from "not worth the same" to "worth the same," in particular when this change occurs after adding a new relation that seemingly has nothing to do with the relation between those two elements. And all these happen because *we* have confounded our experience of equivalence with the masterly defined concept of equivalence (i.e., equivalence relation).

## 7 Conclusions

It took mankind more than 2,000 years to distinguish the notion of equivalence relation from the direct experience of equivalence. And it takes us less than 2 minutes to confound them together in our mathematics classes!

From the vantage point of today's mathematics, an equivalence relation is first and foremost a relation. The latter is defined so that for any two elements (of the underlying set) it is known whether or not *the first is related to the second*. It is then shown that the given relation satisfies certain properties, and because of that we can have different aspects of *our* experience of equivalence. But, the interchangeability of transitivity and F-transitivity, the "missing" symmetry, and for example Euclid's reluctance to use reflexivity show that the standard properties of equivalence relations are more than a simple *indication* of people's experience of equivalence. These properties have been *chosen* to *recreate* certain aspects of this experience. Intriguingly, each one of these chosen properties alone is at odds with our direct experience of equivalence. Ignoring this point could lead to unfortunate pedagogical decisions.

It is standard pedagogical practice to assume that the standard properties of the notion of equivalence relation naturally occur in our experience of equivalence. That is to say, it is assumed that *the* properties of an equivalence relation are embedded in certain situations embodying the idea of equivalence relation. Accordingly, the role of the teacher is to get students to *capture* and *mathematize* the intended properties as the common features of these situations. But, from the outset, some kind of equivalence is directly experienced in the situations and the students have no reason to detach themselves from their experience. As a result, it can be seen that there is no situation paradigmatic enough to help our students to come up with the standard properties of an equivalence relation unless we "read in" the properties. Yet, we may introduce *the* standard properties as a *choice* among others, leading the students "to understand why some organization, some concept, some definition is better than another" (Freudenthal, 1973, p. 418). However, in the case of equivalence relations, it is not easy to practice what we preach.

Each one of the defining properties of the mathematical notion of equivalence relation has been *chosen* for certain mathematical reasons. These reasons can only be revealed in the context of a global picture that took more than 2,000 year to be drawn. It does not seem to be practical (or necessary) to bring all these reasons to the fore in a short time. However, being aware of these reasons enables us to focus our attention on—and draw our students' attention to—the critical features that might otherwise be taken for granted (by us and by our students). Now, it is worth repeating what Marton and Booth (1997, p. 2002) call "the first principle of pedagogy": whenever you fail to get someone to understand something, you have taken something for granted that you should not have taken for granted.

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